

Strong solutions of some one-dimensional SDEs with random and unbounded drifts

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ABSTRACT

In this paper, we are interested in the following one dimensional forward stochastic differential equation (SDE)

$$dX_t = b(t, X_t, \omega)dt + \sigma dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R},$$

where the driving noise B_t is a d -dimensional Brownian motion. The drift coefficient $b : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and can be decomposed into a deterministic and a random part, i.e., $b(t, x, \omega) = b_1(t, x) + b_2(t, x, \omega)$. Assuming that b_1 is of spacial linear growth and b_2 satisfies some integrability conditions, we obtain the existence and uniqueness of a strong solution. The method we use is purely probabilistic and relies on Malliavin calculus. As byproducts, we obtain Malliavin differentiability of the solutions, provide an explicit representation for the Malliavin derivative and prove existence of weighted Sobolev differentiable flows.

KEYWORDS: Malliavin calculus, random drift, measurable drift, compactness criterion, explicit representation, Sobolev differentiable flow.

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1. Introduction

The first main result of the present paper concerns wellposedness of a class of stochastic differential equations of the form

$$dX_t = (b_1(t, X_t) + b_2(t, X_t, \omega)) dt + \sigma dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R} \quad (1.1)$$

when the drift coefficient $b_1 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is merely measurable and of linear growth in the second variable, b_2 is bounded (in x), sufficiently smooth and has bounded derivative (see the assumptions in Section 1.2) and $\sigma \in \mathbb{R}^d$. The driving noise B is the canonical process $B_t(\omega) = \omega_t$ on the canonical space $\Omega := C([0, T], \mathbb{R}^d)$ equipped with the Wiener measure P and the completion $(\mathcal{F})_{t \in [0, T]}$ of the natural filtration of B .

Since the work of Itô [20], it is well known that the SDE (1.1) admits a unique strong solution when the drift b is globally Lipschitz continuous and of linear growth. That is, there exists a unique (up to indistinguishability) square integrable process that is \mathbb{F} -adapted and satisfies (1.1). SDEs are widely applied in stochastic control, in physics, and as a modeling tool, in a number of applied sciences including biology, finance and engineering. Often, the Lipschitz continuity condition is too stringent, as for instance in modeling of switching systems (see e.g. Delong [8] and Heikkilä and Lakshmikantham [19]) or in models of interacting finite (or infinite) particle systems (see e.g. Kondratiev et al. [24] and Alberverio et al. [1]) where the drift b is typically discontinuous. While existence of weak solutions of (1.1) is a direct consequence of Girsanov theory, the construction of strong solutions is usually a delicate matter. Note in addition that in the above mentioned applications existence of a solution X as function of the driving noise (i.e. strong solution) is crucial.

Strong solutions of SDEs with rough coefficients have been extensively studied in the past decades, starting with the seminal works by Zvonkin [41] and Veretennikov [40] and including other important contributions e.g. by Gyöngy and Krylov [17], Gyöngy and Martinez [18]; Krylov and Röckner [25] or Fredrizzi and Flandoli [16]. These works eventually build on the

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analysis of the Kolmogorov partial differential equation associated to the SDE or on pathwise uniqueness arguments and benefit from the Yamada-Watanabe theorem.

Let us further refer to works by Fang and Zhang [12], Fang et al. [13] and more recently Champagnat and Jabin [5] on uniqueness of SDEs. See also Davie [7] for a path by path uniqueness result. A purely probabilistic approach, initiated by Proske [38] and Meyer-Brandis and Proske [32], and further developed by Menoukeu-Pamen et al. [30] rather uses the Malliavin calculus of variations and white noise analysis for the construction of solutions (see also Banos et al. [2]). This method does not rely on pathwise uniqueness arguments, but rather derives it as a consequence of uniqueness in law and strong existence.

As a common feature in the aforementioned works, the drift coefficient b is assumed deterministic (i.e. not depending on ω). This is due to the need to guarantee a Markovian property of the solution, which is paramount for the success of the PDE methods (in finite dimension). As suggested by an anonymous referee, let us mention however that it seems conceivable that, to some extent, the PDE methods could work when the random part b_2 of the drift is seen as a forcing in the equation, but this remains an open question. Random drift also constitute a clear impediment to the success of the probabilistic method due to the "integration by parts estimates" used in several steps of the proofs. Regarding the growth of the drift coefficient, let us mention that to the best of our knowledge, the only works considering SDEs with discontinuous drifts and with linear growth are the articles by Engelbert and Schmidt [11], Nilssen [34] and Menoukeu-Pamen and Mohammed [29]. Unbounded drifts are also treated under $L_t^q(L_x^p)$ -type conditions, see e.g. [14, 15, 25]. All these works consider deterministic coefficients.

In this paper we consider SDEs with coefficients b of the form

$$b(t, x, \omega) = b_1(t, x) + b_2(t, x, \omega) \quad (1.2)$$

for some random non-anticipating stochastic function b_2 , with b_1 a Borel function of spacial linear growth. In particular, b_2 is possibly path-dependent. When b_1 is the gradient of a given function, such SDEs can be seen as dynamics of a diffusion in a random potential see e.g. Kondratiev et al. [24].

SDEs with random coefficients when the drift coefficient does not have the special structure (1.2) have been studied in the literature. For example, Ocone and Pardoux [36] use the generalized Itô-Ventzell formula for anticipating integrands to study a Stratonovich-type SDE, where the initial condition and drift coefficient are allowed to anticipate the future of driving Brownian motion. They show that the Stratonovich-type SDE with anticipating coefficients has a unique non exploding Malliavin differentiable solution. They assume that the initial condition and drift coefficient are Malliavin smooth and the drift is further sublinear with respect to the spatial coordinate and has derivatives of polynomial growth. Assuming that the drift coefficient satisfies a stochastic Lipschitz condition, Kohatsu-Higa et al. [23] show existence and uniqueness of a class of SDE with random coefficients. They do not prove Malliavin differentiability of the solution in their work.

Our method draws from the Malliavin calculus approach of Proske [38] and Menoukeu-Pamen et al. [30] but avoids the use of white noise analysis. In particular, we prove existence and uniqueness of a strong solution and further derive Malliavin differentiability and non-explosion of the solution. The main difficulties in deriving Malliavin smoothness of the solution to the SDE comes from the fact that we do not require any spatial smoothness of the coefficient. The estimates used to derive Malliavin differentiability of solutions allow to further obtain existence of a stochastic flow of dynamical systems driven by (1.1), and this hints at applications to new stochastic transport equations as in the work by Mohammed et al. [33] (see also Flandoli et al. [15], Fedrizzi and Flandoli [14] and Menoukeu-Pamen [28]).

Let us now give a precise statement of the main results of the paper.

1.1. Probabilistic setting

Let $T \in (0, \infty)$ and $d \in \mathbb{N}$ be fixed and consider a probability space (Ω, \mathcal{F}, P) equipped with the completed filtration $(\mathcal{F}_t)_{t \in [0, T]}$ of a d -dimensional Brownian motion B . Throughout the paper, the product $\Omega \times [0, T]$ is endowed with the predictable σ -algebra. Subsets of \mathbb{R}^k , $k \in \mathbb{N}$, are always endowed with the Borel σ -algebra induced by the Euclidean norm $|\cdot|$. The interval $[0, T]$ is equipped with the Lebesgue measure. Unless otherwise stated, all equalities and inequalities

between random variables and processes will be understood in the P -almost sure and $P \otimes dt$ -almost sure sense, respectively. For $p \in [1, \infty]$ and $k \in \mathbb{N}$, denote by $\mathcal{S}^p(\mathbb{R}^k)$ the space of all adapted continuous processes X with values in \mathbb{R}^k such that $\|X\|_{\mathcal{S}^p(\mathbb{R}^k)}^p := E[(\sup_{t \in [0, T]} |X_t|)^p] < \infty$, and by $\mathcal{H}^p(\mathbb{R}^k)$ the space of all predictable processes Z with values in \mathbb{R}^k such that $\|Z\|_{\mathcal{H}^p(\mathbb{R}^k)}^p := E[(\int_0^T |Z_u|^2 du)^{p/2}] < \infty$.

1.2. Main results

In this section, we present the main results of the paper. Refer to the beginning of Section 2 for details regarding Malliavin calculus. Let us consider the following conditions

- (A1) It holds $b = b_1 + b_2$, where the function $b_1 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is Borel measurable and there is $k_1 \geq 0$ such that for all $x \in \mathbb{R}$,

$$|b_1(t, x)| \leq k_1(1 + |x|).$$

The function $b_2 : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is adapted, such that

- a) $b_2(t, \cdot, \omega) \in C^1(\mathbb{R})$ and there exists a random variable $M_2(\omega) \geq 0$ such that

$$|\frac{\partial}{\partial x} b_2(t, x, \omega)| + |b_2(t, x, \omega)| \leq M_2(\omega) \text{ for all } (t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega, \quad (1.3)$$

with

$$b_2^{\text{exp}} := E \left[e^{C|M_2(\omega)|^2} \right] < \infty \quad (1.4)$$

and $C := 48T \max_i \frac{d\sigma_i^2}{(\sigma_1^2 + \dots + \sigma_d^2)^2}$. The constant C depends on k, d and σ (see Lemma 3.1 for its explicit form).

- b) The random variable $b_2(s, x, \cdot)$ is Malliavin differentiable for every $(s, x) \in [0, T] \times \mathbb{R}$ and there exists a process $\tilde{M}_2(t, \omega)$ such that its Malliavin derivative $D_t b_2(s, x, \omega)$ satisfies

$$|D_t b_2(s, x, \omega)| \leq \tilde{M}_2(s, t, \omega) \quad P \otimes dt\text{-a.s., for all } (t, x) \in [0, T] \times \mathbb{R},$$

with

$$b_2^{\text{power}} := \sup_{0 \leq s \leq T} E \left[\left(\int_0^T |\tilde{M}_2(s, t, \omega)|^2 dt \right)^4 \right] < \infty \quad (1.5)$$

and there exist constants $C, \alpha' > 0$ such that

$$E[|D_t b_2(s, x, \omega) - D_{t'} b_2(s, x, \omega)|^4] \leq C|t' - t|^{\alpha'}$$

- (A2) $\sigma \in \mathbb{R}^d$ and $|\sigma|^2 > 0$.

Theorem 1.1. Assume that conditions (A1)-(A2) hold. Then there exists a unique global strong solution $X \in \mathcal{S}^2(\mathbb{R})$ to the SDE

$$dX_t = b(t, X_t, \omega)dt + \sigma dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}. \quad (1.6)$$

The proof is given in Sections 2 and 3. Under the conditions of Theorems 1.1, we show that the unique strong solution of the SDE is Malliavin differentiable and has a Sobolev differentiable flow. Namely, we have

Theorem 1.2. Assume that (A1)-(A2) hold. Let X be the unique strong solution to the SDE (1.6). It holds $X_t \in \mathcal{D}^{1,2}(\mathbb{R})$ for all $t \in [0, T]$.

The proof of this result is given in Section 3 where moment bounds and an explicit representation of the Malliavin derivative is also provided. Denote by $X^{s,x}$ the solution of the SDE (1.1) with initial condition $X_s = x$. As is well-known, the solution $X^{s,x}$ may not belong to the Sobolev space $W^{1,p}(\mathbb{R}, dx)$, $p > 1$. Thus, following the intuition of Mohammed et al. [33] we will show that $X^{s,x}$ belongs to a weighted Sobolev space.

Let $w : \mathbb{R} \rightarrow (0, \infty)$ be a Borel-measurable (weight) function such that

$$\int_{\mathbb{R}} e^{c|x|^2} w(x) dx < \infty$$

for every $c \geq 0$. We denote by $W^{1,p}(\mathbb{R}, w)$ the weighted Sobolev space of functions $u : \mathbb{R} \rightarrow \mathbb{R}$ such that, it holds

$$\|u\|_{1,p,w} := \left(\int_{\mathbb{R}} |u(x)|^p w(x) dx \right)^{1/p} + \left(\int_{\mathbb{R}} |u'(x)|^p w(x) dx \right)^{1/p} < \infty,$$

where u' is the weak derivative of u .

Theorem 1.3. *Assume that (A1)-(A2) hold. Let X^x be the unique solution of (1.1). Let T be small enough. Then for every $p \geq 2$, the map $x \mapsto X_t^x$ belongs to $L^2(\Omega, W^{1,p}(\mathbb{R}, w))$.*

It would be desirable to have multi-dimensional versions of Theorems 1.1 1.2 and 1.3. The main obstacles to the extension of the method presented in this paper to the multi-dimensional case are presented in Remark 2.9. Let us give some examples of drift coefficients satisfying condition (A1).

Example 1.4. The example of a random drift term of the form $b_1(t, x) + \varphi(t, x, B_t)$, where $\varphi : [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz continuous functions (in the second and third variables) seems not to be covered by the existing literature. It is consistent with our assumptions since the Malliavin derivative of $\varphi(t, x, B_t)$ is bounded, and the exponential moment condition (1.4) is satisfied, at least for T small enough, or for arbitrary T when φ is bounded.

A more general example is the path dependent drift case $b(t, \omega, x) := b_1(t, x) + \varphi(t, x, B_{0:t})$, where $B_{0:t}$ denotes the path of B up to t , and $\varphi : [0, T] \times \mathbb{R} \times C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ is bounded and Lipschitz continuous. It follows e.g. from Cheridito and Nam [6, Proposition 3.2] that $\varphi(t, x, B_{0:t})$ has bounded Malliavin derivatives for all t . \diamond

The remainder of the paper is structured as follows: The next section is mainly dedicated to the proof of Theorem 1.1. As a byproduct of our method, we obtain Malliavin differentiability of the solution. In addition, we derive various results concerning the Malliavin derivative of the solution, including moment estimates and a representation in terms of the space-time local time integral. In the appendix we present a few auxiliary results to make the paper self-contained.

2. Existence and Uniqueness

2.1. Some notation

In this section, we prove existence and uniqueness of strong solutions for SDEs. Since Malliavin calculus will play an important role in our arguments, we briefly introduce the spaces of Malliavin differentiable random variables and stochastic processes $\mathcal{D}^{1,p}(\mathbb{R}^k)$ and $\mathcal{L}_a^{1,p}(\mathbb{R}^k)$, $p \geq 1$. For a thorough treatment of the theory of Malliavin calculus we refer to Nualart [35]. Let \mathcal{M} be the class of smooth random variables $\xi = (\xi^1, \dots, \xi^l)$ of the form

$$\xi^i = \varphi^i \left(\int_0^T h_s^{i1} dW_s, \dots, \int_0^T h_s^{in} dW_s \right),$$

where φ^i is in the space $C_{\text{poly}}^\infty(\mathbb{R}^n; \mathbb{R})$ of infinitely continuously differentiable functions whose partial derivatives have polynomial growth, $h^{i1}, \dots, h^{in} \in L^2([0, T]; \mathbb{R}^d)$ and $n \geq 1$. For every ξ in \mathcal{M} let the operator $D = (D^1, \dots, D^d) : \mathcal{M} \rightarrow L^2(\Omega \times [0, T]; \mathbb{R}^d)$ be given by

$$D_t \xi^i := \sum_{j=1}^n \frac{\partial \varphi^i}{\partial x_j} \left(\int_0^T h_s^{i1} dW_s, \dots, \int_0^T h_s^{in} dW_s \right) h_t^{ij}, \quad 0 \leq t \leq T, \quad 1 \leq i \leq l,$$

and the norm $\|\xi\|_{1,p} := (E[|\xi|^p + \int_0^T |D_t \xi|^p dt])^{1/p}$. As shown in Nualart [35], the operator D extends to the closure $\mathcal{D}^{1,p}(\mathbb{R}^l)$ of the set \mathcal{M} with respect to the norm $\|\cdot\|_{1,p}$. A random variable ξ is Malliavin differentiable if $\xi \in \mathcal{D}^{1,p}(\mathbb{R}^l)$ and we denote by $D_t \xi$ its Malliavin derivative. Denote by $\mathcal{L}_a^{1,p}(\mathbb{R}^l)$ the space of processes $Y \in \mathcal{H}^2(\mathbb{R}^l)$ such that $Y_t \in \mathcal{D}^{1,p}(\mathbb{R}^l)$ for all $t \in [0, T]$, the process DY_t admits a square integrable progressively measurable version and

$$\|Y\|_{\mathcal{L}_a^{1,p}(\mathbb{R}^l)}^p := \|Y\|_{\mathcal{H}^p(\mathbb{R}^l)}^p + E \left[\int_0^T \int_0^T |D_r Y_t|^p dr dt \right] < \infty.$$

2.2. Proof of Theorem 1.1

In the whole of this section, we assume that conditions (A1) and (A2) are satisfied. The proof of the Theorem 1.1 is given in 5 steps. In the first step, we show that there exists a process X^x satisfying the SDE (1.6) in the weak sense. That is, there is a Brownian motion \tilde{B} such that (X_t^x, \tilde{B}_t) is a weak solution to the SDE (1.6). Note however that the solution might not be adapted to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$. Let us mention that if X_t is adapted to that filtration then X_t has an explicit representation as a function of B_t (see for example [27, 31]) and for any other stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, T]}, \tilde{P}, \tilde{B})$, the same representation holds with \tilde{B}_t instead of B_t and thus X_t is $(\tilde{\mathcal{F}}_t)_{t \in [0, T]}$ -adapted. The latter indicates that X_t is a strong solution of (1.6).

In the second step, for T small, given a sequence $b_n := b_{1,n} + b_2$ such that $b_{1,n} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $n \geq 1$ are smooth coefficients with compact support and converging a.e. to b_1 , we show using relative compactness (see Lemma 2.3) that for each $0 \leq t \leq T$ the sequence of corresponding strong solutions $(X_t^{x,n})_{n \geq 1}$, of the SDEs

$$dX_t^{x,n} = b_n(t, X_t^{x,n}, \omega) dt + \sigma \cdot dB_t, \quad 0 \leq t \leq T, \quad X_0^{x,n} = x \in \mathbb{R}, \quad n \geq 1, \quad (2.1)$$

is relatively compact in $L^2(P; \mathbb{R})$. Let us mention that existence of a unique strong solution to the SDE (2.1) is guaranteed by [36, Theorem 1.1], see also [21].

In step 3, we show that for each $0 \leq t \leq T$ the above sequence $(X_t^{x,n})_{n \geq 1}$ converges weakly to $E[X_t^x | \mathcal{F}_t]$ in the space $L^2(\Omega, \mathcal{F}_t, P)$. This with step 2 allow to deduce that $(X_t^{x,n})_{n \geq 1}$ converges strongly to $E[X_t^x | \mathcal{F}_t]$ in the space $L^2(\Omega, \mathcal{F}_t, P)$. We also obtain from step 2 that $E[X_t^x | \mathcal{F}_t]$ is Malliavin smooth, see Subsection 3.

In step 4, we prove that X_t^x is \mathcal{F}_t -measurable by showing that $E[X_t^x | \mathcal{F}_t] = X_t^x$. The proof is completed by showing uniqueness.

In the last step, we use a pasting argument to show that the result holds for all $T > 0$. In fact, the linear growth assumption on b_1 and integrability assumption on b_2 ensure by the use of Gronwall's lemma that if the solution exists on a small interval then it does not explode. Hence the main task in this step is to show that $E[|D_s X_t^{x,n}|^2] \leq C$, uniformly in n for $0 \leq s \leq t \leq T$.

2.2.1. Weak existence. The following result can be seen as a slight generalization of a result by V.E. Beneš, compare [3, 22]. Therein (and throughout the paper) we denote by $\mathcal{E}(\int q dB)$ the Doléan-Dade exponential

$$\mathcal{E}\left(\int q dB\right)_t := \exp\left(\int_0^t q_u dB_u - \frac{1}{2} \int_0^t |q_u|^2 du\right)$$

for a given progressive process q such that $\int_0^T |q_u|^2 du < \infty$.

Lemma 2.1. *Let u be given by*

$$u_i = \frac{\sigma_i}{\sigma_1^2 + \dots + \sigma_d^2} b. \quad (2.2)$$

Then the process $Z := \mathcal{E}\left(\int u(r, \sigma \cdot B_r, \omega) dB_r\right)$ is a martingale.

Proof. The proof follows from that of [22, Theorem 2.1]. We know that the process Z_t is a non negative local martingale and thus a supermartingale such that $E[Z_t] \leq 1$ for any $0 \leq t \leq T$. Using the same argument as in the proof of [22, Theorem 2.1], one obtains the result by applying Gronwall's lemma provided that $E[|M_2(\omega)|^4] < \infty$. The later is true by assumption. \square

The next lemma ensures weak existence, it is a simple adaptation of [21, Proposition 5.3.6].

Lemma 2.2. *The SDE (1.6) admits a weak solution X_t^x .*

Proof. Let (Ω, \mathcal{F}, P) be a probability space on which a d -dimensional Brownian motion \hat{B} is given, and set $X_t^x := x + \sigma \cdot \hat{B}_t$, $0 \leq t \leq T$. By (A1), it follows from Lemma 2.1 (see also [3, 4, 22]) that the process $\mathcal{E}\left(\int u(r, X_r^x, \omega) d\hat{B}_r\right)$ defines an equivalent probability measure Q given by

$$\frac{dQ}{dP} := \mathcal{E}\left(\int u(r, X_r^x, \omega) d\hat{B}_r\right)_T.$$

In addition, Girsanov's theorem asserts that $B_t = \hat{B}_t - \int_0^t u(r, X_r^x, \omega) dr$ is a Q -Brownian motion. Therefore,

$$X_t^x = x + \int_0^t \sigma \cdot u(s, X_s^x, \omega) ds + \sigma \cdot B_t \quad Q\text{-a.s.}, \quad 0 \leq t \leq T, \quad (2.3)$$

showing that (X^x, B) is a weak solution to the SDE (1.6) on the probability space (Ω, \mathcal{F}, Q) . \square

2.2.2. Approximation and compactness. Let $b_n = b_{1,n} + b_2$ be such that $b_{1,n} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $n \geq 1$ are smooth coefficients with compact support and converging a.e. to b_1 . Denote by $X_t^{x,n}$ the unique strong solution to the SDE (2.1) with drift b_n . The following result is key to the compactness argument.

Lemma 2.3. *If $T \in (0, \infty)$ is small enough, the strong solution $X_t^{x,n}$ of the SDE (2.1) satisfies*

$$E[|D_t X_s^{x,n} - D_{t'} X_s^{x,n}|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{power}) |t - t'|^\alpha$$

for all $0 \leq t' \leq t \leq T$ and some $\alpha = \alpha(s) > 0$. Moreover,

$$\sup_{0 \leq t \leq T} E[|D_t X_s^{x,n}|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{power}),$$

where the function $\mathcal{C}(\cdot, \cdot, \cdot) : [0, \infty)^3 \rightarrow [0, \infty)$ is continuous and increasing in each components, b_2^{power} defined in (1.5) and

$$\|\tilde{b}_1\|_\infty := \text{ess sup} \left\{ \frac{|b_1(t, z)|}{1 + |z|} : t \in [0, T], z \in \mathbb{R} \right\}. \quad (2.4)$$

The combination of Lemma 2.3 and Corollary A.3 yields the following result:

Corollary 2.4. *For each $0 \leq t \leq T$, with T small enough, the sequence $(X_t^{x,n})_{n \geq 1}$, is relatively compact in $L^2(P; \mathbb{R})$.*

Proof (of Lemma 2.3). Since the Brownian motions are independent, applying the chain-rule formula for the Malliavin derivatives in the direction of the i^{th} Brownian motion (see e.g. [35]) gives

$$D_t^i X_s^{x,n} = \sigma_i + \int_t^s D_t^i b_2(u, X_u^{x,n}, \omega) du + \int_t^s \left\{ b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega) \right\} D_t^i X_u^{x,n} du, \quad i = 1, \dots, d \text{ } P\text{-a.s.} \quad (2.5)$$

for all $t \leq s \leq T$. Here $b'_{1,n}(t, x) := \frac{\partial}{\partial x} b_{1,n}(t, x)$ and $b'_2(t, x, \omega) := \frac{\partial}{\partial x} b_2(t, x, \omega)$ are the spatial derivatives of $b_{1,n}$ and b_2 , respectively. Solving (2.5) explicitly gives

$$D_t^i X_s^{x,n} = e^{\int_t^s \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \left(\int_t^s D_t^i b_2(u, X_u^{x,n}, \omega) e^{-\int_t^u \{b'_{1,n}(r, X_r^{x,n}) + b'_2(r, X_r^{x,n}, \omega)\} dr} du + \sigma_i \right). \quad (2.6)$$

Let $0 \leq t' \leq t \leq s \leq T$. Using the above representation, we have

$$\begin{aligned} & D_{t'}^i X_s^{x,n} - D_t^i X_s^{x,n} \\ &= e^{\int_{t'}^s \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \left(\int_{t'}^s D_{t'}^i b_2(u, \omega) e^{-\int_{t'}^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_{t'}^u b'_{1,n}(r, X_r^{x,n}) dr} du + \sigma_i \right) \\ &\quad - e^{\int_t^s \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \left(\int_t^s D_t^i b_2(u, X_u^{x,n}, \omega) e^{-\int_t^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_t^u b'_{1,n}(r, X_r^{x,n}) dr} du + \sigma_i \right) \\ &= \sigma_i e^{\int_t^s \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \left(e^{\int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} - 1 \right) \\ &\quad + \int_{t'}^s D_{t'}^i b_2(u, X_u^{x,n}, \omega) e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} du - \int_t^s D_t^i b_2(u, \omega) e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} du \\ &= \sigma_i e^{\int_t^s \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \left(e^{\int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} - 1 \right) \\ &\quad + \int_{t'}^t D_{t'}^i b_2(u, X_u^{x,n}, \omega) e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} du \\ &\quad + \int_t^s (D_{t'}^i b_2(u, X_u^{x,n}, \omega) - D_t^i b_2(u, X_u^{x,n}, \omega)) e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} du \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (2.7)$$

Set

$$u_{i,n} = \frac{\sigma_i}{\sigma_1^2 + \dots + \sigma_d^2} b_n. \quad (2.8)$$

Then using Hölder inequality repeatedly, we have

$$\begin{aligned}
E[I_1^2] &= \sigma_i^2 E \left[e^{2 \int_t^s b'_2(u, X_u^{x,n}, \omega) du} e^{2 \int_t^s b'_{1,n}(u, X_u^{x,n}) du} \left(e^{\int_t^t b'_{1,n}(u, X_u^{x,n}) du} e^{\int_t^t b'_2(u, X_u^{x,n}, \omega) du} - 1 \right)^2 \right] \\
&\leq \sigma_i^2 E \left[e^{6 \int_t^s b'_2(u, X_u^{x,n}, \omega) du} \right]^{\frac{1}{3}} E \left[e^{6 \int_t^s b'_{1,n}(u, X_u^{x,n}) du} \right]^{\frac{1}{3}} E \left[\left(e^{\int_t^t b'_{1,n}(u, X_u^{x,n}) du} e^{\int_t^t b'_2(u, X_u^{x,n}, \omega) du} - 1 \right)^6 \right]^{\frac{1}{3}} \\
&= \sigma_i^2 E \left[e^{6 \int_t^s b'_2(u, X_u^{x,n}, \omega) du} \right]^{\frac{1}{3}} E \left[\mathcal{E} \left(\int_0^T u_n(r, x + \sigma \cdot B_r, \omega) dB_r \right) e^{6 \int_t^s b'_{1,n}(u, x + \sigma \cdot B_u) du} \right]^{\frac{1}{3}} \\
&\quad \times E \left[\left(e^{\int_t^t b'_{1,n}(u, X_u^{x,n}) du} e^{\int_t^t b'_2(u, X_u^{x,n}, \omega) du} - 1 \right)^6 \right]^{\frac{1}{3}} \\
&\leq C \sigma_i^2 E \left[e^{2 \sum_{i=1}^d \int_0^T u_{i,n}(r, x + \sigma \cdot B_r, \omega) dB_r - 2 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{6}} E \left[e^{4 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{12}} \\
&\quad \times E \left[e^{24 \int_t^s b'_{1,n}(u, x + \sigma \cdot B_u) du} \right]^{\frac{1}{12}} E \left[\left(e^{\int_t^t b'_{1,n}(u, X_u^{x,n}) du} e^{\int_t^t b'_2(u, X_u^{x,n}, \omega) du} - 1 \right)^6 \right]^{\frac{1}{3}} \\
&= I_{1,1}^{\frac{1}{6}} + I_{1,2}^{\frac{1}{12}} + I_{1,3}^{\frac{1}{12}} + I_{1,4}^{\frac{1}{3}}. \tag{2.9}
\end{aligned}$$

It follows from the Girsanov theorem applied to the martingale $2 \sum_{i=1}^d \int_0^\cdot u_{i,n}(r, x + \sigma \cdot B_r, \omega) dB_r$ that the first term $I_{1,1}$ in (2.9) is equal to one. Next, we wish to use conditions on b_n and thus u_n to show that the second term is finite for T small enough. Using Hölder inequality, we have

$$E \left[e^{4 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right] \leq \prod_{i=1}^d E \left[e^{12d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{d}} \leq \prod_{i=1}^d E \left[e^{12d \int_0^T \frac{\sigma_i^2}{(\sigma_1^2 + \dots + \sigma_d^2)^2} b_n^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{d}}.$$

Let us focus on each component of the above product. Using the condition on b_n , Hölder inequality successively and the independence of the Brownian motion, we get

$$\begin{aligned}
E \left[e^{12d \int_0^T \frac{\sigma_i^2}{(\sigma_1^2 + \dots + \sigma_d^2)^2} b_n^2(r, x + \sigma \cdot B_r, \omega) dr} \right] &\leq E \left[e^{24c_{d,\sigma} \int_0^T (k^2 |1+x+\sigma \cdot B_r|^2 + |M_2(\omega)|^2) dr} \right] \\
&\leq E \left[e^{48c_{d,\sigma} \int_0^T k^2 (1+|x+\sigma \cdot B_r|)^2 dr} \right]^{\frac{1}{2}} \times E \left[e^{48c_{d,\sigma} T |M_2(\omega)|^2} \right]^{\frac{1}{2}} \\
&\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} E \left[e^{48c_{d,\sigma} k^2 \int_0^T |\sigma \cdot B_r|^2 dr} \right]^{\frac{1}{2}} \\
&\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} E \left[e^{24c_{d,\sigma} k^2 T \sup_{0 \leq t \leq T} |\sigma \cdot B_t|^2} \right]^{\frac{1}{2}} \\
&\leq C e^{48c_{d,\sigma} k^2 T (1+|x|)^2} E \left[e^{48c_{d,\sigma} k^2 T \sum_{i=1}^d \sup_{0 \leq t \leq T} |\sigma_i \cdot B_t^i|^2} \right]^{\frac{1}{2}} \\
&\leq C_{T,d,\sigma,M_2} e^{48c_{d,\sigma} k^2 T (1+|x|)^2} \prod_{i=1}^d E \left[e^{48c_{d,\sigma} k^2 T \sup_{0 \leq t \leq T} |\sigma_i \cdot B_t^i|^2} \right]^{\frac{1}{2}}. \tag{2.10}
\end{aligned}$$

In the above, $c_{d,\sigma} := \frac{d\sigma_i^2}{(\sigma_1^2 + \dots + \sigma_d^2)^2}$. Now, using exponential expansion and the Doob maximal inequality, we have

$$\begin{aligned}
E \left[e^{48c_{d,\sigma} k^2 T \sup_{0 \leq t \leq T} |\sigma_i \cdot B_t^i|^2} \right] &= 1 + \sum_{p=1}^{\infty} \frac{(48c_{d,\sigma} \sigma_i^2 k^2 T)^p}{p!} E \left[\sup_{0 \leq t \leq T} |B_t|^{2p} \right] \\
&\leq 1 + \sum_{p=1}^{\infty} \frac{(48k^2 d T)^p}{p!} \left(\frac{2p}{2p-1} \right)^2 \frac{(2p)!}{2^p \cdot p!} T^p. \tag{2.11}
\end{aligned}$$

The inequality comes from the fact that $\frac{d\sigma_i^2\sigma_i^2}{(\sigma_1^2+\dots+\sigma_d^2)^2} \leq d$. Next, applying the ratio test to the series $\sum_p a_p$ with $a_p := \frac{(48k_1^2dT^2)^p}{p!} \left(\frac{2p}{2p-1}\right)^2 \frac{(2p)!}{2^p \cdot p!}$ for $p \geq 1$, one can easily show that the series converge for example for

$$T \leq T_1 := \frac{1}{4\sqrt{3}dk_1^2}. \quad (2.12)$$

Hence the second term in (2.9) is finite for small T .

Now, we turn to $I_{1,4}$ in (2.9). Using power and exponential expansion, we get by linearity of the expectation and Hölder inequality

$$\begin{aligned} & E \left[\left(e^{\int_{t'}^t b'_{1,n}(u, X_u^{x,n}) du} e^{\int_{t'}^t b'_2(u, X_u^{x,n}, \omega) du} - 1 \right)^6 \right] \\ &= E \left[e^{6 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} - 6e^{5 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} + 15e^{4 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} \right. \\ &\quad \left. - 20e^{3 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} + 15e^{2 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} - 6e^{\int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du} + 1 \right] \\ &= E \left[\sum_{q=1}^{\infty} \frac{\left(6 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] - 6E \left[\sum_{q=1}^{\infty} \frac{\left(5 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] \\ &\quad + 15E \left[\sum_{q=1}^{\infty} \frac{\left(4 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] - 20E \left[\sum_{q=1}^{\infty} \frac{\left(3 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] \\ &\quad + 15E \left[\sum_{q=1}^{\infty} \frac{\left(2 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] - 6E \left[\sum_{q=1}^{\infty} \frac{\left(\int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right)^q}{q!} \right] \\ &\leq E \left[\sum_{q=1}^{\infty} \frac{\left| 6 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] + 6E \left[\sum_{q=1}^{\infty} \frac{\left| 5 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] \\ &\quad + 15E \left[\sum_{q=1}^{\infty} \frac{\left| 4 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] + 20E \left[\sum_{q=1}^{\infty} \frac{\left| 3 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] \\ &\quad + 15E \left[\sum_{q=1}^{\infty} \frac{\left| 2 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] + 6E \left[\sum_{q=1}^{\infty} \frac{\left| \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q}{q!} \right] \\ &\leq J_1 + J_2 + J_3 + J_4 + J_5 + J_6. \end{aligned} \quad (2.13)$$

Since $\sigma \cdot B_t \sim N(0, t \sum_{i=1}^d \sigma_i^2)$ and has independent increments, it follows from Proposition B.1 that each term in (2.13) is bounded by $\mathcal{C}(T, \|\tilde{b}_1\|_{\infty}, |x|)|t - t'|$, where $\mathcal{C}(\|\tilde{b}_1\|_{\infty}, |x|)$ is a continuous function depending on $\|\tilde{b}\|_{\infty}, x, \|\sigma\|^2$ and T . More specifically, let us focus on J_1 only since the bounds for the other terms follow in a similar way. Using dominated

convergence theorem, Hölder inequality and Girsanov theorem, we have

$$\begin{aligned}
J_1 &= \sum_{q=1}^{\infty} \frac{E \left[\left| 6 \int_{t'}^t \{b'_{1,n}(u, X_u^{x,n}) + b'_2(u, X_u^{x,n}, \omega)\} du \right|^q \right]}{q!} \\
&\leq \sum_{q=1}^{\infty} \frac{12^p E \left[\left| \int_{t'}^t b'_{1,n}(u, X_u^{x,n}) du \right|^q \right]}{q!} + \sum_{q=1}^{\infty} \frac{12^p E \left[\left| \int_{t'}^t b'_2(u, X_u^{x,n}, \omega) du \right|^q \right]}{q!} \\
&\leq \sum_{q=1}^{\infty} \frac{12^p E \left[\mathcal{E} \left(\int_0^T u_n(r, x + \sigma \cdot B_r, \omega) dB_r \right) \left| \int_{t'}^t b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^q \right]}{q!} + \sum_{q=1}^{\infty} \frac{12^p |t' - t|^q E \left[C_T^p |M_2(\omega)|^q \right]}{q!} \\
&\leq E \left[e^{4 \sum_{i=1}^d \int_0^T u_{i,n}(r, \omega, x + \sigma \cdot B_r) dB_r^i - 8 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, \omega, x + \sigma \cdot B_r) dr} \right]^{\frac{1}{4}} E \left[e^{6 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, \omega, x + \sigma \cdot B_r) dr} \right]^{\frac{1}{4}} \\
&\quad \times \sum_{q=1}^{\infty} \frac{12^p E \left[\left| \int_{t'}^t b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^{2q} \right]^{\frac{1}{2}}}{q!} + |t' - t|^{\frac{1}{2}} \sum_{q=1}^{\infty} \frac{12^p |t' - t|^{q-\frac{1}{2}} E \left[C_T^p |M_2(\omega)|^q \right]}{q!} \\
&\leq C \sum_{q=1}^{\infty} \frac{12^p E \left[\left| \int_{t'}^t b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^{2q} \right]^{\frac{1}{2}}}{q!} + \frac{C}{\sqrt{T}} E \left[\exp \{12 C_T M_2(\omega)\} \right] |t' - t|^{\frac{1}{2}}, \tag{2.14}
\end{aligned}$$

where the first bound comes from Girsanov theorem applied to the martingale $2 \sum_{i=1}^d \int_0^T u_{i,n}(r, \omega, x + \sigma \cdot B_r) dB_r$ and similar computations as in the case of $I_{1,2}$. Now by Proposition B.1,

$$\begin{aligned}
\sum_{q=1}^{\infty} \frac{E \left[\left(\int_{t'}^t \sqrt{12} b'_{1,n}(u, x + \sigma \cdot B_u) du \right)^{2q} \right]^{\frac{1}{2}}}{q!} &\leq \left(\sum_{q=1}^{\infty} \frac{C_{\sigma,k}^q (1 + |x|^q) \sqrt{q!} (t - t')^{q/2}}{q!} \right) \\
&\leq \left(\sum_{q=1}^{\infty} \frac{C_{\sigma,k}^q (1 + |x|^q) (t - t_0)^{\frac{q-1}{2}}}{\sqrt{q!}} \right) |t - t'|^{1/2} \tag{2.15}
\end{aligned}$$

for some positive constant C_{σ} .

Multiplying the numerator and the denominator of each term in the series by 2^q and using Cauchy-Schwartz inequality yields:

$$\begin{aligned}
\sum_{q=1}^{\infty} \frac{E \left[\left| \int_{t'}^t \sqrt{12} b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^{2q} \right]}{q!} &\leq \left(\sum_{q=1}^{\infty} \frac{2^q C_{\sigma,k}^q (1 + |x|^q) (t - t_0)^{\frac{q-1}{2}}}{\sqrt{q!} 2^q} \right) |t - t'|^{1/2} \\
&\leq \left(\sum_{q=1}^{\infty} \frac{2^q C_{\sigma,k}^{2q} (1 + |x|)^{2q} (t - t_0)^{(q-1)}}{q!} \right)^{1/2} \left(\sum_{q=1}^{\infty} \frac{1}{4^q} \right)^{1/2} |t - t'|^{1/2} \\
&\leq C \left(\sum_{q=1}^{\infty} \frac{2^q C_{\sigma,k}^{2q} (1 + |x|)^{2q} T^{(q-1)}}{q!} \right)^{1/2} |t - t'|^{1/2} \\
&\leq \frac{C}{\sqrt{T}} \exp \{C_{\sigma,k} T (1 + |x|)^2\} |t - t'|^{1/2}. \tag{2.16}
\end{aligned}$$

Similarly, it can be proved that $E \left[e^{16 \int_t^s b'_{1,n}(u, x + \sigma \cdot B_u) du} \right]$ is bounded by $\frac{C}{\sqrt{T}} \exp \{C_{\sigma,k} T (1 + |x|)^2\} |t - s|^{1/2}$. Therefore there exists a constant C depending on σ such that

$$E[I_1^2] \leq \frac{C}{\sqrt{T}} \exp \{C_{\sigma,k} T (1 + |x|)^2\} |t - t'|^{1/2}.$$

Repeated application of the Hölder inequality yields

$$\begin{aligned}
E[I_2^2] &= E\left[\left(\int_{t'}^t D_{t'}^i b_2(u, X_u^{n,x}, \omega) e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} du\right)^2\right] \\
&\leq E\left[\int_{t'}^t \left(D_{t'}^i b_2(u, X_u^{n,x}, \omega)\right)^2 du \left(\int_{t'}^t e^{-4\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u 4b'_{1,n}(r, X_r^{x,n}) dr} du\right)^{1/2}\right] (t-t')^{1/2} \\
&\leq E\left[\left(\int_{t'}^t \left(D_{t'}^i b_2(u, X_u^{n,x}, \omega)\right)^2 du\right)^2\right]^{\frac{1}{2}} E\left[\int_{t'}^t e^{-4\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u 4b'_{1,n}(r, X_r^{x,n}) dr} du\right]^{\frac{1}{2}} (t-t')^{1/2} \\
&\leq C|t-t'|^{1/2} E\left[\left(\int_{t'}^t \left(\tilde{M}_2(u, t', \omega)\right)^2 du\right)^2\right]^{\frac{1}{2}} E\left[\int_0^T e^{-4\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u 4b'_{1,n}(r, X_r^{x,n}) dr} du\right]^{\frac{1}{2}} \\
&\leq C_{T, b_2^{\text{power}}} |t-t'|^{1/2} E\left[\int_0^T e^{-4\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} e^{-\int_s^u 4b'_{1,n}(r, X_r^{x,n}) dr} du\right]^{\frac{1}{2}}. \tag{2.17}
\end{aligned}$$

Again, using Girsanov transform and Hölder inequality, similar reasoning as before gives

$$E[I_2^2] \leq C_{T, b_2^{\text{power}}} \frac{C}{\sqrt{T}} \exp\{C_{\sigma, k} T(1 + |x|)^2\} |t-t'|^{\frac{1}{2}}$$

for $0 \leq t' \leq t \leq T$, with T small enough.

As for I_3 , once more repeated use of Cauchy-Schwartz inequality and assumption (A1) give the existence of a constant C that may change from one line to the other such that

$$\begin{aligned}
E[I_3^2] &= E\left[\left(\int_t^s \left(D_{t'}^i b_2(u, X_u^{x,n}, \omega) - D_t^i b_2(u, X_u^{x,n}, \omega)\right) e^{-\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} e^{-\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} du\right)^2\right] \\
&\leq CE\left[\int_t^s |D_{t'}^i b_2(u, X_u^{x,n}, \omega) - D_t^i b_2(u, X_u^{x,n}, \omega)|^4 du\right]^{1/2} E\left[\int_t^s e^{-4\int_s^u b'_{1,n}(r, X_r^{x,n}) dr} e^{-4\int_s^u b'_2(r, X_r^{x,n}, \omega) dr} du\right]^{1/2} \\
&\leq CE\left[\int_t^s e^{-8\int_{t'}^u b'_{1,n}(r, X_r^{x,n}) dr} du\right]^{\frac{1}{4}} E\left[\int_t^s e^{-8\int_{t'}^u b'_2(r, X_r^{x,n}, \omega) dr} du\right]^{1/4} |t-t'|^\alpha. \tag{2.18}
\end{aligned}$$

Once again, using Girsanov theorem and the linear growth condition on the drift b_1 , one can show that the expectations $E\left[\int_t^s e^{-\int_s^u 8b'_{1,n}(r, X_r^{x,n}) dr} du\right]$ is bounded by $\frac{C}{\sqrt{T}} \exp\{C_{\sigma, k} T(1 + |x|)^2\}$. Moreover, the assumptions on b_2 insure that the two last integral terms on the right hand side of (2.18) are bounded by C . Therefore, there exists $\alpha > 0$ such that

$$E[I_3^2] \leq C \frac{C}{\sqrt{T}} \exp\{C_{\sigma, k} T(1 + |x|)^2\} |t-t'|^\alpha \tag{2.19}$$

for $0 \leq t' \leq t \leq T_1$ with T_1 small enough.

Combining (2.9)-(2.19), there exists a function $\mathcal{C} = \mathcal{C}(k, |x|^2, b_2^{\text{power}}) > 0$ depending on k, b_2 and $|x|^2$ such that

$$E[|D_t X_s^{x,n} - D_{t'} X_s^{x,n}|^2] \leq \mathcal{C}(k, |x|^2, b_2^{\text{power}}) |t-t'|^\alpha$$

for $0 \leq t' \leq t \leq T$ with T small enough ($T \leq 1 \wedge T_1$) and $\alpha' = \min(\alpha, 1/2)$. Thus the first part of the Lemma is shown. Taking $t' > s$ above, $D_{t'} X_s^{x,n} = 0$, which implies

$$\sup_{0 \leq t \leq T} E [|D_t X_s^{x,n}|^2] \leq C(k, |x|^2, b_2^{\text{power}}).$$

This proves the lemma. \square

2.2.3. Weak convergence to the weak solution. In this step, we show that for each $0 \leq t \leq T$ the above sequence $(X_t^{x,n})_{n \geq 1}$ converges weakly to $E[X_t^x | \mathcal{F}_t]$ in the space $L^2(\Omega, P; \mathcal{F}_t)$.

Lemma 2.5. Assume $b_2^{\text{exp}} < \infty$ and Ω is the canonical space. Choose the sequence $b_{1,n} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $n \geq 1$ as before, and let $(X_t^{x,n})_{n \geq 1}$ be the corresponding strong solutions to the SDE (2.1). Then for each $0 \leq t \leq T$, and each function $h : \mathbb{R} \rightarrow \mathbb{R}$ of polynomial growth, the sequence $(h(X_t^{x,n}))_{n \geq 1}$ is uniformly bounded in $L^2(\Omega, P; \mathcal{F}_t)$ and converges weakly to $E[h(X_t^x) | \mathcal{F}_t]$ in this space.

Proof. Let us first show that $(h(X_t^{x,n}))_{n \geq 1}$ is bounded in $L^2(\Omega, P; \mathcal{F}_t)$. In fact, using Girsanov transform, Hölder inequality and the fact that $(1 + |z|^p)e^{-|z|^2/2s}$ can be bounded by $C_p e^{-\frac{|z|^2}{2p+1}s}$, where C_p is a constant depending on p , we have

$$\begin{aligned} \sup_n E [|h(X_t^{x,n})|^2] &\leq E \left[e^{2 \sum_{i=1}^d \int_0^T u_{i,n}(r, x + \sigma \cdot B_r, \omega) dB_r^i - 2 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{2}} \\ &\quad \times E \left[e^{2 \sum_{i=1}^d \int_0^T u_{i,n}^2(r, x + \sigma \cdot B_r, \omega) dr} \right]^{\frac{1}{4}} E [|h(x + \sigma \cdot B_t)|^4]^{\frac{1}{4}} \\ &\leq C E [|h(x + \sigma \cdot B_t)|^4]^{\frac{1}{4}} \\ &= C \left(\frac{1}{\sqrt{2\pi t} \|\sigma\|^2} \int_{\mathbb{R}} |h(x + z)|^4 e^{-|z|^2/2t\|\sigma\|^2} dz \right)^{\frac{1}{4}} \\ &\leq \frac{C \|\sigma\|^2}{(2\pi t \|\sigma\|^2)^{\frac{1}{8}}} \left(|x|^4 \int_{\mathbb{R}} e^{-\frac{|z|^2}{2t\|\sigma\|^2}} dz + \int_{\mathbb{R}} e^{-\frac{|z|^2}{2^5 t \|\sigma\|^2}} dz \right)^{\frac{1}{4}} < \infty. \end{aligned} \quad (2.20)$$

The constant $C_{\|\sigma\|^2}$ above depends only on $\|\sigma\|^2$ and $|x|$.

To show that $(h(X_t^{x,n}))_{n \geq 1}$ converges weakly to $E[h(X_t^x) | \mathcal{F}_t]$ in $L^2(\Omega, P; \mathcal{F}_t)$, first notice that the space

$$\left\{ \mathcal{E} \left(\int_0^T \dot{\varphi}_u dB_u \right) : \varphi \in C_b^1([0, T], \mathbb{R}^d) \right\}$$

is a dense subspace of $L^2(\Omega, P)$. Here $C_b^1([0, T], \mathbb{R}^d)$ is the space of bounded continuous differentiable functions on $[0, T]$ and with values in \mathbb{R}^d and $\dot{\varphi}$ is the derivative of φ . Hence, it is enough to show that $\left(h(X_t^{x,n}) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right)_n$ converges to $E[h(X_t^x) | \mathcal{F}_t] \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right)$ in expectation. Since Ω is a Wiener space, we know from the Cameron-Martin theorem, see e.g. [39], that for every h measurable,

$$E \left[h(X_t^x) \mathcal{E} \left(\int_0^T \dot{\varphi}_u dB_u \right) \right] = \int_{\Omega} h(X_t^x(\omega + \varphi)) dP(\omega). \quad (2.21)$$

Let $\varphi \in C_b^1([0, T], \mathbb{R}^d)$. For every n , the process $\tilde{X}^{x,n}$ given by $\tilde{X}_t^{x,n}(\omega) := X_t^{x,n}(\omega + \varphi)$ solves the SDE

$$d\tilde{X}_t^{x,n} = (b_{1,n}(t, \tilde{X}_t^{x,n}) + \tilde{b}_2(t, \tilde{X}_t^{x,n}, \omega) + \sigma \dot{\varphi}_t) dt + \sigma d\tilde{B}_t \quad (2.22)$$

where $\tilde{b}_2(t, x, \omega) := b_2(t, x, \omega + \varphi)$. To see this, let $H \in L^2(\Omega, P)$ and apply (2.21) and the fact that $X^{x,n}$ solves the SDE (2.1) to get

$$\begin{aligned} E[\tilde{X}_t^{x,n} H] &= E \left[X_t^{x,n} H(\omega - \varphi) \mathcal{E} \left(\int_0^T \dot{\varphi}(u) dB_u \right) \right] \\ &= E \left[\left(x + \int_0^t b_1(u, X_u^{x,n}) + b_2(u, X_u^{x,n}, \omega) du + \sigma B_t \right) H(\omega - \varphi) \mathcal{E} \left(\int_0^T \dot{\varphi} dB \right) \right] \\ &= E \left[\left(x + \int_0^t b_1(u, X_u^{x,n}(\omega + \varphi)) + b_2(u, X_u^{x,n}(\omega + \varphi), \omega + \varphi) du + \sigma B_t(\omega + \varphi) \right) H \right] \\ &= E \left[\left(x + \int_0^t b_1(u, \tilde{X}_u^{x,n}(\omega)) + \tilde{b}_2(u, \tilde{X}_u^{x,n}(\omega), \omega) + \sigma \dot{\varphi} du + \sigma B_t(\omega) \right) H \right], \end{aligned}$$

where the last equality follows by the fact that $B_t(\omega + \varphi) = B_t(\omega) + \varphi$, since B is the canonical process. This proves the claim. Since X^x satisfies the SDE (without been adapted to the filtration (\mathcal{F}_t)), with respect to a probability measure Q which is equivalent to P , see the proof of Lemma 2.2, the above arguments show that $\tilde{X}^x(\omega) := X^x(\omega + \varphi)$ satisfies

$$d\tilde{X}_t^x = (b_1(t, \tilde{X}_t^x) + \tilde{b}_2(t, \tilde{X}_t^{x,n}, \omega) + \sigma \dot{\varphi}_t) dt + \sigma dB_t \quad P\text{-a.s.} \quad (2.23)$$

Now, put

$$\tilde{u}_{i,n} := \frac{\sigma_i}{\sigma_1^2 + \dots + \sigma_d^2} (b_{1,n} + \tilde{b}_2) =: b_{1,n}^{\sigma_i} + \tilde{b}_2^{\sigma_i} \quad \text{and} \quad \tilde{u}_i = \frac{\sigma_i}{\sigma_1^2 + \dots + \sigma_d^2} (b_1 + \tilde{b}_2) =: b_1^{\sigma_i} + \tilde{b}_2^{\sigma_i}. \quad (2.24)$$

It follows by Girsanov theorem that

$$\begin{aligned} E \left[h(X_t^{x,n}) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right] - E \left[h(X_t^x) | \mathcal{F}_t \right] \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) &= E \left[\left(h(X_t^{x,n}) - h(X_t^x) \right) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right] \\ &= E \left[h(x + \sigma \cdot B_t) \left(\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r \right\} dB_r \right) - \mathcal{E} \left(\int_0^T \left\{ \tilde{u}(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r \right\} dB_r \right) \right) \right]. \end{aligned} \quad (2.25)$$

Using the fact that $|e^a - e^b| \leq |e^a + e^b| |a - b|$, the Hölder inequality and Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned} &E \left[h(X_t^{x,n}) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right] - E \left[h(X_t^x) | \mathcal{F}_t \right] \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \\ &\leq CE \left[h(x + \sigma \cdot B_t)^2 \right]^{\frac{1}{2}} E \left[\left(\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r \right\} dB_r \right) + \mathcal{E} \left(\int_0^T \left\{ \tilde{u}(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r \right\} dB_r \right) \right)^4 \right]^{\frac{1}{4}} \\ &\quad \times \left\{ E \left[\left(\int_0^T \left(\tilde{u}_n(r, x + \sigma \cdot B_r, \omega) - \tilde{u}(r, x + \sigma \cdot B_r, \omega) \right) dB_r \right)^4 \right] \right. \\ &\quad \left. + E \left[\left(\int_0^T \left\| \tilde{u}_n(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}(t) \right\|^2 - \left\| \tilde{u}(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r \right\|^2 dr \right)^4 \right] \right\}^{\frac{1}{4}} \\ &= I_1 \times I_{2,n} \times (I_{3,n} + I_{4,n}). \end{aligned} \quad (2.26)$$

That I_1 is finite was proved in the computations leading to (2.20). Observe that

$$\begin{aligned} \mathcal{E}\left(\int_0^T \{\tilde{u}_n(r, x + \sigma \cdot B_r, \omega) + \dot{\varphi}_r\} dB_r\right) &= \mathcal{E}\left(\int_0^T b_{1,n}^\sigma(r, x + \sigma \cdot B_r, \omega) dB_r\right) \mathcal{E}\left(\int_0^T \tilde{b}_2^\sigma dB_r\right) \mathcal{E}\left(\int_0^T \dot{\varphi} dB_r\right) \\ &\times \exp\left(-\int_0^T \dot{\varphi}_r b_{1,n}^\sigma(r, x + \sigma \cdot B_r, \omega) - \dot{\varphi}_r \tilde{b}_2^\sigma - \tilde{b}_2^\sigma b_{1,n}^\sigma(r, x + \sigma \cdot B_r, \omega) dr\right). \end{aligned}$$

Thus, $I_{2,n}$ is bounded by similar argument as in the proof of Lemma 2.3 since $\dot{\varphi}$ is bounded. Using the dominated convergence theorem, we get that $I_{3,n}$ and $I_{4,n}$ converge to 0 as n goes to infinity. \square

The following result is a corollary of the compactness result given by Lemma 2.3 and Corollary 2.4.

Proposition 2.6. *For any fixed $t \in [0, T]$, with T small and $x \in \mathbb{R}$, the sequence $(X_t^{n,x})_{n \geq 1}$ of strong solutions to the SDE (2.1) converges strongly in $L^2(\Omega, P; \mathbb{R})$ to $E[X_t^x | \mathcal{F}_t]$.*

Proof. Observe that by the compactness criteria for each t , there exists a subsequence $(X_t^{x,n_k})_{k \geq 1}$ that converges strongly in $L^2(\Omega, P)$. From the previous lemma, we get by setting $h(x) = x, x \in \mathbb{R}$ that $(X_t^{x,n_k})_{n \geq 1}$ converges weakly to $E[X_t^x | \mathcal{F}_t]$ in $L^2(\Omega, P)$ and therefore by the uniqueness of the limit there exists a subsequence n_k such that $(X_t^{x,n_k})_{n \geq 1}$ converges strongly to $E[X_t^x | \mathcal{F}_t]$ in $L^2(\Omega, P)$. The convergence then holds for the entire sequence by uniqueness of limit. Indeed, by contradiction, suppose that for some t , there exist $\epsilon > 0$ and a subsequence $n_l, l \geq 0$ such that

$$\|X_t^{x,n_l} - E[X_t^x | \mathcal{F}_t]\|_{L^2(\Omega, P)} \geq \epsilon.$$

We also know by the compactness criteria that there exists a further subsequence of $n_m, m \geq 0$ of $n_l, l \geq 0$ such that

$$X_t^{x,n_{n_m}} \text{ converges to } \tilde{X}_t \text{ in } L^2(\Omega, P) \text{ as } m \text{ goes to } \infty.$$

Nevertheless, $(X_t^{x,n_k})_{n \geq 1}$ converges weakly to $E[X_t^x | \mathcal{F}_t]$ in $L^2(\Omega, P)$, and hence by the uniqueness of the limit, we have

$$\tilde{X}_t = E[X_t^x | \mathcal{F}_t].$$

Since

$$\|X_t^{x,n_{n_m}} - E[X_t^x | \mathcal{F}_t]\|_{L^2(\Omega, P)} \geq \epsilon,$$

this is a contradiction. \square

2.2.4. Adaptedness of the weak solution and uniqueness. Finally, we show that the weak solution X_t^x is $(\mathcal{F}_t)_{t \in [0, T]}$ -adapted and unique.

Theorem 2.7. *The weak solution X_t^x to the SDE (1.6) is \mathcal{F}_t -measurable.*

Proof. Let us first show that X_t^x is \mathcal{F}_t -measurable. Let h be a globally Lipschitz continuous function. By Proposition 2.6, there exists a subsequence $n_k, k \geq 0$, such that $h(X_t^{x,n_k})$ converges to $h(E[X_t^x | \mathcal{F}_t])$ P -a.s. as k goes to infinity. Moreover, we know that $h(X_t^{x,n_k})$ converges to $E[h(X_t^x) | \mathcal{F}_t]$ weakly in $L^2(\Omega, P)$ as k goes to infinity. We get from the uniqueness of the limit that

$$h(E[X_t^x | \mathcal{F}_t]) = E[h(X_t^x) | \mathcal{F}_t] \text{ } P\text{-a.s.}$$

Since the above holds for any arbitrary globally Lipschitz continuous function, it follows that X_t^x is \mathcal{F}_t -measurable. \square

Proposition 2.8. *The SDE (1.6) satisfies the pathwise uniqueness property.*

Proof. Let X_t^x and \tilde{X}_t^x be two solutions to the SDE (1.6). For $\varphi \in C_b^1([0, T], \mathbb{R}^d)$, we have

$$E \left[X_t^x \mathcal{E} \left(\int_0^T \dot{\varphi}_u dB_u \right) \right] = \int_{\Omega} X_t^x(\omega + \varphi) dP(\omega), \quad (2.27)$$

where, as shown in the course of the proof of Lemma 2.5, $X_t^x(\omega + \varphi)$ satisfies the SDE (2.22). Similarly $\tilde{X}_t^x(\omega + \varphi)$ satisfies the same SDE. Thus, since the drift is of linear growth, it follows that $(X_t^x(\omega + \varphi), B)$ and $(\tilde{X}_t^x(\omega + \varphi), B)$ have the same distribution. In fact, using that the distributions P^x and \tilde{P}^x of X^x and \tilde{X}^x , respectively are equal to P (see the construction in [21, Proposition 3.6]) it follows that by assumptions on b_2 and the linear growth of b_1 that $\int_0^T |b_1(t, X_t^x) + (\sigma \dot{\varphi}_t + b_2(t, X_t^x, \omega + \varphi))|^2 dt < \infty$ P^x -a.s. The same holds if X^x is replaced by \tilde{X}^x and P^x by \tilde{P}^x . Thus, a simple adaptation of the proof of [21, Proposition 3.10] shows that $(X_t^x(\omega + \varphi), B)$ and $(\tilde{X}_t^x(\omega + \varphi), B)$ have the same distribution. Hence, for all t, φ , we have $E \left[\tilde{X}_t^x \mathcal{E} \left(\int_0^T \dot{\varphi}_u dB_u \right) \right] = E \left[X_t^x \mathcal{E} \left(\int_0^T \dot{\varphi}_u dB_u \right) \right]$, from which pathwise uniqueness follows. \square

2.2.5. Global existence. Since the small time T_1 for which the solution exists does not depend on the initial condition (see (2.12)) one can use a standard pasting argument to show that the solution exists for all time $T > 0$. In addition using the linear growth condition on b_1 and the integrability condition on b_2 , it follows from Gronwall lemma that the unique solution does not explode.

The proof of Theorem 1.1 is now complete. \square

Remark 2.9. For the solvability of the SDE (1.1) in the multi-dimensional case, the main technical hurdle is the use of the explicit solution of the (random) ODE (2.5) in the proof of Lemma 2.3, which does not carry-over to the multi-dimensions. An alternative approach could be to employ the iteration procedure used in [29]. But in that case, due to the randomness of the drift, the estimations do not reduce to the applications of the key integration by parts estimate in Proposition 2.8 which in turn was an important argument in the proof of compactness of the approximating sequence. \blacklozenge

3. Malliavin differentiability

3.1. Differentiability of the strong solution

In this subsection, we show that the unique strong solution of the SDE (1.6) constructed in the previous subsection is Malliavin differentiable and we derive a representation formula of the Malliavin derivative. The proof Malliavin differentiability for small time interval follows directly from Lemma 2.3. In order to control the Malliavin derivative of the process on arbitrary time intervals, we need the following result which is a variant of Fernique theorem whose proof is similar to that of [29, Lemma 2.6].

Lemma 3.1. *Let $t_0 \in [0, 1]$ and let $\eta : \Omega \rightarrow \mathbb{R}$ be a \mathcal{F}_{t_0} -measurable random variable independent of the P -augmented filtration generated by the Brownian motion B . Let $b : [0, 1] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that $b = b_1 + b_2$ satisfies (A1), with b_1 a smooth coefficient with compact support satisfying a global linear growth condition. Denote by $X_t^{t_0, \eta}$ the unique strong solution (if it exists) to the SDE (1.6) starting at η and with drift coefficient b . Then we can find a positive number δ_0 independent of t_0 and η (but may depend on $\|b_1\|_{\infty}$) such that*

$$E \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 \right\} \leq C_1 E \exp \{ C_2 \delta_0 |\eta|^2 \}, \quad (3.1)$$

where C_1, C_2 are positive constants independent of η , but may depend on k_1 and b_2 . Moreover, C_1 may depend on δ_0 . Furthermore, if the right hand side of (3.1) is finite then the above expectation is finite.

Proof. We have the following almost sure equality

$$X_t^{t_0, \eta} = \eta + \int_{t_0}^t \left(b_1(u, X_u^{t_0, \eta}) + b_2(u, X_u^{t_0, \eta}, \omega) \right) du + \sigma \cdot (B_t - B_{t_0}), \quad t_0 \leq t \leq 1. \quad (3.2)$$

Successive application of Hölder's inequality to (3.2) yields

$$\begin{aligned} |X_t^{t_0, \eta}|^2 &\leq 4|\eta|^2 + 4 \left| \int_{t_0}^t b(u, X_u^{t_0, \eta}) du \right|^2 + 4 \left| \int_{t_0}^t b_2(u, X_u^{t_0, \eta}, \omega) du \right|^2 + 4|\sigma|^2 |B_t - B_{t_0}|^2 \\ &\leq 4|\eta|^2 + 4 \left(\int_{t_0}^t k_1(1 + |X_u^{t_0, \eta}|) du \right)^2 + 4 \left| \int_{t_0}^t b_2(u, X_u^{t_0, \eta}, \omega) du \right|^2 + 4|\sigma|^2 |B_t - B_{t_0}|^2 \\ &\leq 4|\eta|^2 + 8k_1^2(t - t_0) \int_{t_0}^t \{1 + |X_u^{t_0, \eta}|^2\} du + 4(t - t_0) \int_{t_0}^t |b_2(u, X_u^{t_0, \eta}, \omega)|^2 du + 4|\sigma|^2 |B_t - B_{t_0}|^2 \\ &\leq 4|\eta|^2 + 8k_1^2(t - t_0)^2 + 8k_1^2(t - t_0) \int_{t_0}^t |X_u^{t_0, \eta}|^2 du + 4(t - t_0)^2 C_T^2 |M_2(\omega)|^2 + 4|\sigma|^2 |B_t - B_{t_0}|^2, \text{ a.s.} \end{aligned} \quad (3.3)$$

Take the supremum on both sides of (3.3) and multiply by δ_0 to get

$$\begin{aligned} \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 &\leq 4\delta_0 |\eta|^2 + 8k_1^2 \delta_0 + 8k_1^2 \int_{t_0}^1 \delta_0 \sup_{0 \leq u \leq s} |X_u^{t_0, \eta}|^2 ds \\ &\quad + 4\delta_0(t - t_0)^2 C_T^2 |M_2(\omega)|^2 + 4|\sigma|^2 \delta_0 \sup_{t_0 \leq t \leq 1} |B_t - B_{t_0}|^2, \text{ a.s.} \end{aligned} \quad (3.4)$$

Applying Gronwall's lemma to (3.4), we have

$$\delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 \leq \left\{ 4\delta_0 |\eta|^2 + 8k_1^2 \delta_0 + 4\delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 + 4\delta_0 |M_2(\omega)|^2 \right\} e^{8k_1^2}, \text{ a.s.} \quad (3.5)$$

Now, set $C_2 := 4e^{8k_1^2}$. Then taking exponential on both sides of (3.5), we have

$$\begin{aligned} \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 \right\} &\leq \exp \{ 2C_2 \delta_0 k^2 \} \times \exp \{ \delta_0 C_2 |\eta|^2 \} \times \exp \left\{ C_2 \delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 \right\} \\ &\quad \times \exp \{ C_2 \delta_0 |M_2(\omega)|^2 \}, \quad P\text{-a.s.} \end{aligned} \quad (3.6)$$

Taking expectations in the above and using once more Hölder inequality, we get

$$\begin{aligned} E \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{t_0, \eta}|^2 \right\} &\leq \exp \{ 2C_2 k^2 \delta_0 \} \cdot E \left[\exp \{ 3C_2 \delta_0 |\eta|^2 \} \right]^{\frac{1}{3}} \times E \left[\exp \left\{ 3C_2 \delta_0 \|\sigma\|^2 \sup_{t_0 \leq u \leq t} |B_u - B_{t_0}|^2 \right\} \right]^{\frac{1}{3}} \\ &\quad \times E \left[\exp \{ 3C_2 \delta_0 |M_2(\omega)|^2 \} \right]^{\frac{1}{3}}. \end{aligned} \quad (3.7)$$

The result follows provided that we find δ_0 independent of η and t_0 such that

$$E \left[\exp \left\{ 3C_2 \delta_0 |\sigma|^2 \sup_{t_0 \leq u \leq 1} |B_u - B_{t_0}|^2 \right\} \right] < \infty. \quad (3.8)$$

The estimate (3.8) is obtained from the Fernique theorem. In fact, one can show that for $\delta_0 < \min(\frac{1}{12dC_2|\sigma|^2}, \frac{1}{C_2})$, the result holds (see for example [29, Lemma 2.6] for details). From this (3.7) yields

$$E \exp \left\{ \delta_0 \sup_{t_0 \leq t \leq 1} |X_t^{n, t_0, \eta}|^2 \right\} \leq C_1 E e^{C_2 \delta_0 |\eta|^2}. \quad (3.9)$$

Note that C_1, C_2 and δ_0 are independent of η and t_0 (but may depend on $\|\tilde{b}_1\|_\infty$ and $|\sigma|^2$). Thus (3.1) is valid for the above choice of δ_0 . \square

Next, using Lemma 3.1, we prove that under the conditions of Theorem 1.1 the Malliavin derivative is bounded in the $L^2(\Omega, P)$ norm.

Proof (of Theorem 1.2). First recall that by the second part of Lemma 2.3, the sequence of strong solutions of the SDE (2.1) satisfies

$$\sup_{0 \leq t \leq T} \sup_{n \geq 1} E [|D_t X_s^{x, n}|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{\text{power}}).$$

Since the Malliavin derivative is a closable operator (see e.g. [35, Exercise 1.2.3]), it follows from Proposition 2.6 and Theorem 2.7 that X_t^x is Malliavin smooth.

It remains to prove integrability of the derivative. This is done by induction. Choose δ_0 as in Lemma 3.1 and define

$$\tau := \frac{\delta_0}{64d\sqrt{2}k^2} \leq T_1,$$

let $s_i = i\tau$ and $x_i := X_{s_i}^{x, n, 0}, i \geq 1$. It follows from the previous argument that the result is valid for $0 \leq t \leq s_1$.

Assume that there exists a Malliavin differentiable solution $\{X_t, 0 \leq t \leq s_m\}$. Set t such that $s_m \leq t < s_{m+1}$. Let $(X_t^{x, n})_{n \geq 1}$ be the approximating sequence defined by (2.1) it satisfies the a.s. relation

$$X_t^{x_m, n} = X_{s_m}^n + \int_{s_m}^t b_n(u, X_u^{x_m, n}, \omega) du + \sigma(B_t - B_{s_m}), \quad s_m \leq t \leq s_{m+1}.$$

Using the chain-rule for the Malliavin derivatives, we have component wise

$$D_s^i X_t^{x_m, n} = \begin{cases} D_s^i X_{s_m}^n + \int_{s_m}^t \left(\{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\} D_s^i X_u^{x_m, n} + D_s^i b_2(t, X_u^{x_m, n}, \omega) \right) du & \text{if } s \leq s_m \\ \sigma_i + \int_s^t \left(\{b'_{1,n}(u, X_u) + b'_2(u, X_u^{x_m, n}, \omega)\} D_s^i X_u^{x_m, n} + D_s^i b_2(t, X_u^{x_m, n}, \omega) \right) du & \text{if } s > s_m \end{cases}, \quad (3.10)$$

P -a.s. for all $0 \leq s \leq t$ and solving explicitly gives

$$D_s^i X_t^{x_m, n} = \begin{cases} e^{\int_{s_m}^t \{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\} du} \left(\int_{s_m}^t D_s^i b_2(u, X_u^{x_m, n}) e^{-\int_{s_m}^u \{b'_{1,n}(r, X_r^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\} dr} du + D_s^i X_{s_m}^n \right) & \text{if } s \leq s_m \\ e^{\int_t^s \{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\} du} \left(\int_t^s D_s^i b_2(u, X_u^{x_m, n}) e^{-\int_t^u \{b'_{1,n}(r, X_r^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\} dr} du + \sigma_i \right) & \text{if } s > s_m \end{cases} \quad (3.11)$$

P -a.s. for all $0 \leq s \leq t$. Let us first focus on $D_s X_t^{x_m, n}$, when $s \leq s_m$. Using Hölder inequality, we have

$$\begin{aligned}
E[|D_s^i X_t^n|^2] &\leq 2E\left[e^{\int_{s_m}^t 2\{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\}du} \left(\int_{s_m}^t D_s^i b_2(u, X_u^{x_m, n}, \omega) e^{-\int_{s_m}^u \{b'_{1,n}(r, X_r^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\}dr} du\right)^2\right] \\
&\quad + 2E\left[e^{2\int_{s_m}^t \{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\}du} |D_s^i X_{s_m}^n|^2\right] \\
&\leq 2E\left[e^{\int_{s_m}^t 2\{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\}du} \left(\int_{s_m}^t D_s^i b_2(u, X_u^{x_m, n}, \omega) e^{-\int_{s_m}^u \{b'_{1,n}(r, X_r^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\}dr} du\right)^2\right] \\
&\quad + 4E\left[e^{\int_{s_m}^t 4b'_{1,n}(u, X_u^{x_m, n})du} |D_s^i X_{s_m}^n|^2\right] + E\left[e^{4\int_{s_m}^t b'_2(r, X_r^{x_m, n}, \omega)du} |D_s^i X_{s_m}^n|^2\right] \\
&= 2E\left[e^{\int_{s_m}^t 2\{b'_{1,n}(u, X_u^{x_m, n}) + b'_2(u, X_u^{x_m, n}, \omega)\}du} \left(\int_{s_m}^t D_s^i b_2(u, X_u^{x_m, n}, \omega) e^{-\int_{s_m}^u \{b'_{1,n}(r, X_r^{x_m, n}) + b'_2(r, X_r^{x_m, n}, \omega)\}dr} du\right)^2\right] \\
&\quad + 4E\left[E\left[e^{\int_{s_m}^t 4b'_{1,n}(u, X_u^{x_m, n})du} \mathcal{F}_{s_m}\right] |D_s^i X_{s_m}^n|^2\right] + 4E\left[e^{8\int_{s_m}^t b'_2(r, X_r^{x_m, n}, \omega)du}\right]^{\frac{1}{2}} E\left[|D_s^i X_{s_m}^n|^4\right]^{\frac{1}{2}} \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let us now consider the conditional expectation part in I_2 . As before, using Girsanov theorem and Hölder inequality, we have

$$\begin{aligned}
&E\left[e^{\int_{s_m}^t 4b'_{1,n}(u, X_u^{x_m, n})du} \mathcal{F}_{s_m}\right] \\
&\leq E\left[\mathcal{E}\left(\int_{s_m}^t u_n(u, \omega, X_{s_m}^n + \sigma \cdot (B_u - B_{s_m}))dB_u\right) e^{\int_{s_m}^t 4b'_{1,n}(u, X_{s_m}^n + \sigma \cdot (B_u - B_{s_m}))du} \mathcal{F}_{s_m}\right] \\
&\leq E\left[e^{2\sum_{i=1}^d \int_{s_m}^{s_m+1} u_{i,n}(r, \omega, X_{s_m}^n + \sigma \cdot (B_r - B_{s_m}))dB_r - 2\sum_{i=1}^d \int_{s_m}^{s_m+1} u_{i,n}^2(r, \omega, X_{s_m}^n + \sigma \cdot (B_r - B_{s_m}))dr} \mathcal{F}_{s_m}\right]^{\frac{1}{2}} \\
&\quad \times E\left[e^{6\sum_{i=1}^d \int_{s_m}^{s_m+1} u_{i,n}^2(r, \omega, X_{s_m}^n + \sigma \cdot (B_r - B_{s_m}))dr} \mathcal{F}_{s_m}\right]^{\frac{1}{4}} E\left[e^{16\int_{s_m}^t b'_{1,n}(u, X_{s_m}^n + \sigma \cdot (B_u - B_{s_m}))du} \mathcal{F}_{s_m}\right]^{\frac{1}{4}}. \tag{3.12}
\end{aligned}$$

By Girsanov theorem applied to the martingale $2\int_{s_m}^{s_m+1} b_n(r, X_{s_m}^n + B_r - B_{s_m}, \omega)dB_r$ the first term of the right hand side is equal to one. Since $b_{1,n}$ is of spatial linear growth, $B_r - B_{s_m}$ is independent of \mathcal{F}_{s_m} and $X_{s_m}^n$ is \mathcal{F}_{s_m} -measurable, it follows from the Hölder inequality, the exponential expansion and the choice of τ that there exists a constant $C > 0$ such that

$$E\left[e^{16\int_{s_m}^{s_m+1} u_{i,n}^2(r, X_{s_m}^n + \sigma \cdot (B_u - B_{s_m}), \omega)dr} \mathcal{F}_{s_m}\right]^{\frac{1}{4}} \leq C e^{6k_1^2 \tau (1 + |X_{s_m}^n|^2)}. \tag{3.13}$$

Next let us consider the last term. We can show as in [29, Proposition 4.10 and (4.28)] that there exists a constant $C > 0$ such that

$$E\left[e^{16\int_{s_m}^t b'_{1,n}(u, X_{s_m}^n + \sigma \cdot (B_u - B_{s_m}))du} \mathcal{F}_{s_m}\right]^{\frac{1}{4}} \leq e^{\tau C k_1 |X_{s_m}^n|}. \tag{3.14}$$

Combining (3.12)-(3.14) and using Hölder inequality, we get

$$\begin{aligned}
I_2 &\leq CE\left[e^{6k_1^2 \tau (1 + |X_{s_m}^n|^2)} e^{\tau C k_1 |X_{s_m}^n|} |D_s^i X_{s_m}^n|^2\right] \\
&\leq CE\left[e^{24k_1^2 \tau (1 + |X_{s_m}^n|^2)}\right]^{\frac{1}{4}} E\left[e^{4\tau C k_1 |X_{s_m}^n|}\right]^{\frac{1}{4}} E\left[|D_s^i X_{s_m}^n|^4\right]^{\frac{1}{2}}. \tag{3.15}
\end{aligned}$$

Let us notice that one can show as in the proof of Lemma 2.3 that there exists $C > 0$ such that $E[|D_s^i X_{s_1}^n|^p] \leq C$ for $p \geq 1$ and by induction hypothesis it follows that $E[|D_s^i X_{s_m}^n|^p] \leq C$ for $p = 4$. The choice of τ combined with Lemma 3.1 ensures

that one can find $C > 0$ such that $E[e^{24k_1^2\tau(1+|X_{s_m}^n|^2)}]^{1/4} \leq Ce^{C_2\delta_0|x|^2}$. Furthermore, using successive approximation, one can show that $E[e^{4\tau Ck_1|X_{s_m}^n|}] \leq C_1e^{C_2k_1|x|}$, where C_1 depends on b_2 and C_2 is a positive constant. This can be shown as in the proof of Lemma 3.1 using the Gronwall lemma and the probability distribution of the Brownian motion.

The case $s > s_m$ is similar (and easier) since the term $D_s^i X_{s_m}^m$ is not involved, it is replaced by the constant σ .

Since $b_2^{\text{power}} < \infty$, using the Hölder inequality, the term I_1 can also be bounded using similar arguments as above. Thus the Malliavin derivative of the approximating sequence $(X_t^{x,n})_{n \geq 1}$ has a uniform bound on $[0, T]$ which does not depend on n . Therefore, $D_s X_t \in L^2(\Omega, P; \mathbb{R}^d)$ for $0 \leq s \leq t \leq T$. \square

Remark 3.2. Let us observe that if b_1 is globally bounded then it follows from the condition on b_2 and Girsanov theorem that Lemma 2.3 holds for all T . Therefore, we do not need the above argument and the Malliavin differentiability of the solution directly follows from the compactness argument. \blacklozenge

The following result gives estimates on the Malliavin derivative of the solution.

Theorem 3.3. *Assume that the conditions of Theorem 1.1 are satisfied. Then, the Malliavin derivative of the unique strong solution to the SDE (1.6) satisfies:*

$$E[|D_t X_s^x - D_{t'} X_s^x|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{\text{power}})|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq T_1$, $\alpha = \alpha(s) > 0$ and

$$\sup_{0 \leq t \leq T_1} E[|D_t X_s^x|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{\text{power}}),$$

where $C : [0, \infty)^3 \rightarrow [0, \infty)$ is continuous and increasing in each component and can differ from line to line, T_1 is small, and $\|\tilde{b}_1\|_\infty$ defined in (2.4).

Proof. Let $b_{1,n}$ be a sequence of smooth functions with compact support converging a.e. to b_1 and such that $b_{1,n}$ satisfies a uniform global linear growth; that is $\|\tilde{b}_{1,n}\|_\infty \leq \|\tilde{b}_1\|_\infty$, see (2.4) for definition. Let X^n be the solution of the SDE (2.1) with drift $b_n := b_{1,n} + b_2$. Then by Lemma 2.3, for every $n \in \mathbb{N}$ it holds

$$E[|D_t X_s^n - D_{t'} X_s^n|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{\text{power}})|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq T_1$, $\alpha = \alpha(s) > 0$ and

$$\sup_{0 \leq t \leq T_1} E[|D_t X_s^n|^2] \leq C(\|\tilde{b}_1\|_\infty, |x|^2, b_2^{\text{power}}),$$

where the r.h.s. do not depend on n . In particular, $D_t^i X_s^n$ is bounded in $L^2(\Omega, P)$. Thus, it follows from Corollary 2.4 and [35, Lemma 1.2.3] that (up to a subsequence) $(D_t^i X_s^n)_n$ converges to $D_t^i X_s$ in the weak topology of $L^2(\Omega, P)$. Since the function $L^2 \ni A \mapsto E[|A|^2]$ is convex and lower semicontinuous, it is weakly lower semicontinuous. Thus, taking limits in n on both sides above gives the results. \square

3.2. Representation and moment bounds for the Malliavin derivative

In this subsection we give an explicit representation of the Malliavin derivative DX^x of the solution X^x of the SDE (1.6). To that end, we will assume that the random part b_2 of the drift does not depend on x . Such representation can be very useful to derive results concerning DX^x . The representation we obtain will be given in terms of the time-space local time studied in details in [9].

In order to define the local time-space integral with respect to $L^{X^x}(t, z)$, we first start by introducing the space $(\mathcal{H}^x, \|\cdot\|_x)$ of functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ with the norm

$$\|f\|_x := 2 \left(\int_0^1 \int_{\mathbb{R}} f^2(s, z) \exp\left(-\frac{|z-x|^2}{2s}\right) \frac{ds dz}{\sqrt{2\pi s}} \right)^{\frac{1}{2}} + \int_0^1 \int_{\mathbb{R}} |z-x| |f(s, x)| \exp\left(-\frac{|z-x|^2}{2s}\right) \frac{ds dz}{s\sqrt{2\pi s}}.$$

See for example [9]. Endowed with this normed, $(\mathcal{H}^x, \|\cdot\|_x)$ is a Banach space. It follows from [2, Lemma 2.7 and Definition 2.8] that the local time-space integral of $f \in \mathcal{H}^x$ with respect to $L^{X^x}(t, z)$ is well defined and we have

$$\int_0^t \int_{\mathbb{R}} f(s, z) L^{X^x}(ds, dz) = \int_0^T \int_{\mathbb{R}} f(s, z) I_{[0, t]}(s) L^{X^x}(ds, dz). \quad (3.16)$$

Let us point out that as already observed in [2, Remark 2.9], functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ of spacial linear growth uniformly in t belong to \mathcal{H}^x and thus the above local space-time integral exists for $x \in \mathbb{R}$.

We will also need the following representation which will play a key role in our argument (see for example [2, Lemma 2.11]). Let $f \in \mathcal{H}^x$ be Lipschitz continuous in space. Then for all $t \in [0, T]$, it holds

$$\int_0^t f'(s, X_s^x) ds = - \int_0^t \int_{\mathbb{R}} f(s, z) L^{X^x}(ds, dz). \quad (3.17)$$

Moreover, the local time-space integral of $f \in \mathcal{H}^0$ admits the decomposition (see the proof of [9, Theorem 3.1])

$$\int_0^t \int_{\mathbb{R}} f(s, z) L^{B^x}(ds, dz) = \int_0^t f(s, B_s^x) dB_s + \int_{T-t}^T f(T-s, \widehat{B}_s) d\widetilde{W}_s - \int_{T-t}^T f(T-s, \widehat{B}_s) \frac{\widehat{B}_s}{T-s} ds,$$

$0 \leq t \leq T$, a.s., B^x is the Brownian motion started at x and \widehat{B} is the time-reversed Brownian motion, that is

$$\widehat{B}_t := B_{T-t}, \quad 0 \leq t \leq T. \quad (3.18)$$

Further, the process \widetilde{W}_t , $0 \leq t \leq T$, is an independent Brownian motion with respect to the filtration $\mathcal{F}_t^{\widehat{B}}$ generated by \widehat{B}_t , and satisfies:

$$\widetilde{W}_t = \widehat{B}_t - B_T + \int_t^T \frac{\widehat{B}_s}{T-s} ds. \quad (3.19)$$

We are now ready to give an explicit representation of the Malliavin derivative of the unique strong solution to the SDE (1.6) in terms of a local time integral.

Theorem 3.4. *Assume that the conditions of Theorem 1.1 are satisfied. Assume in addition that b_2 does not depend on x and that*

$$\sup_{0 \leq s \leq T} E \left[\left(\int_0^T |D_s b_2(t, \omega + \varphi)|^2 dt \right)^2 \right] < \infty \quad (3.20)$$

for every $\varphi \in C_b^1([0, T], \mathbb{R}^d)$. For every $0 \leq s \leq t \leq T$, the i -th component of the Malliavin derivative of the unique strong solution to the SDE (1.6) admits the following representation:

$$D_t^i X_s^x = e^{\int_t^s \int_{\mathbb{R}} b_1(u, z) L^{X^x}(ds, dz)} \left(\int_t^s D_t^i b_2(u, \omega) e^{-\int_t^u \int_{\mathbb{R}} b_1(r, z) L^{X^x}(dr, dz)} du + \sigma_i \right), \quad i = 1, \dots, d, \quad (3.21)$$

$L^{X^x}(ds, dz)$ is the integration with respect to the time-space local time of X^x .

Before proving the above theorem we will need some auxiliary results. The next one generalizes [2, Lemma A.2] to the case where the integrand is of spatial linear growth.

Remark 3.5. When the drift b_2 does not depend on x , the SDE (1.1) can be solved without use of Malliavin calculus. In fact, by a change of measures, the SDE reduces to an equation with deterministic drift, driven by a new Brownian motion (solved e.g. in [29]). Then, one can conclude using the fact that the filtration of the new Brownian motion is smaller than the initial filtration. \blacklozenge

Example 3.6. Let $\alpha \in \mathcal{H}^4(\mathbb{R}^d)$ be such that $b_2(t, \omega) := \int_0^t \alpha_s dB_s$ satisfies the moment condition (1.4) (which is automatically satisfied when α is deterministic), and α Hölder-continuous in t , and Malliavin differentiable with $(D_s \alpha_t)_t \in \mathcal{H}^4(\mathbb{R}^d)$. Then, it follows by Nualart [35, Proposition 1.3.4] that b_2 is Malliavin differentiable, and

$$D_s b_2(t) = \alpha_s 1_{\{s \leq t\}} + \int_s^t D_s \alpha_r dB_r. \quad (3.22)$$

Thus, by Burkholder-Davis-Gundy inequality, it holds

$$\sup_{0 \leq s \leq T} E \left[\left(\int_0^T |D_s b_2(t, \omega)|^2 dt \right)^4 \right] \leq \|\alpha\|_{\mathcal{H}^4(\mathbb{R}^d)}^4 + \sup_{0 \leq s \leq T} \|D_s \alpha\|_{\mathcal{H}^4(\mathbb{R}^d)}^4 < \infty,$$

which shows that $b_2^{\text{power}} < \infty$. \blacklozenge

Remark 3.7. Let us notice that using Cameron-Martin-Girsanov theorem, one can show that the bound (3.20) holds if, for $\varepsilon > 0$ small, we have

$$\sup_{0 \leq s \leq T} E \left[\int_0^T |D_s b_2(t, \omega)|^{4+\varepsilon} dt \right] < \infty. \quad (3.23) \quad \blacklozenge$$

Lemma 3.8. Let $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be of spacial linear growth uniformly in t . Then for every $t \in [0, T]$, $k \in \mathbb{R}$ and subset $K \subset \mathbb{R}$, it holds

$$\sup_{x \in K} E \left[\exp \left(k \int_0^t \int_{\mathbb{R}} f(s, z) L^{B^x}(ds, dz) \right) \right] < \infty,$$

provided that T is small enough. In the above, $L^{B^x}(ds, dz)$ is the integration with respect to the time-space local time of B^x .

Proof. It follows from (3.2) and the Hölder inequality that

$$E \left[\exp \left(k \int_0^t \int_{\mathbb{R}} f(s, z) L^{B^x}(ds, dz) \right) \right] \quad (3.24)$$

$$\begin{aligned} &\leq E \left[\exp \left(2k \int_0^t f(s, B_s^x) dB_s^x \right) \right]^{\frac{1}{2}} \times E \left[\exp \left(4k \int_{T-t}^T f(T-s, B_{T-s}^x) d\widetilde{W}_s^x \right) \right]^{\frac{1}{4}} \\ &\times E \left[\exp \left(-4k \int_{T-t}^T f(T-s, B_{T-s}^x) \frac{B_{T-s}^x}{T-s} ds \right) \right]^{\frac{1}{4}} = I_1 \times I_2 \times I_3. \end{aligned} \quad (3.25)$$

Let us consider I_1 . Using Hölder inequality, we have

$$E \left[\exp \left(2k \int_0^t f(s, B_s^x) dB_s^x \right) \right] \leq E \left[\mathcal{E} \left(4k \int_0^T f(s, B_s^x) dB_s^x \right) \right]^{\frac{1}{2}} E \left[\exp \left(8k^2 \int_0^T f^2(s, B_s^x) ds \right) \right]^{\frac{1}{2}}.$$

The Girsanov theorem applied to the martingale $2k \int_0^t f(s, B_s^x) dB_s^x$ yields that the first term in (2.9) is equal to one. Similar arguments as in the proof of Lemma 2.3 (i.e. using power series expansion of the exponential function) enables to conclude that the second term above is finite for small T .

Next we wish to study the boundedness I_3 . It was already shown in [2, Lemma A.2] that

$$E \left[\exp \left(k \int_0^T \frac{|B_{T-s}|}{T-s} ds \right) \right] < \infty.$$

Hence to show the boundedness of I_3 , it suffices to show that

$$E \left[\exp \left(k \int_0^T \frac{|B_{T-s}|^2}{T-s} ds \right) \right] < \infty$$

for T small enough. Indeed, using exponential expansion, and the Hölder inequality, we have

$$\begin{aligned} E \left[\exp \left(k \int_0^T \frac{|B_{T-s}|^2}{T-s} ds \right) \right] &= \sum_{n=1}^{\infty} \frac{1}{n!} E \left[\left(k \int_0^T \frac{|B_{T-s}|^2}{T-s} ds \right)^n \right] \leq \sum_{n=1}^{\infty} \frac{k^n}{n!} \int_0^T \frac{E|B_{T-s}|^{2n}}{(T-s)^n} ds \times T^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{k^n}{n!} \frac{(2n)!}{2^n n!} T \times T^{n-1} = \sum_{n=1}^{\infty} (Tk)^n \frac{(2n)!}{2^n (n!)^2}. \end{aligned}$$

Using once more the ratio test, one deduces that the above sum is finite for small T . Combining arguments for the bounds of I_1 and I_3 enables to conclude that I_2 is bounded as well. \square

Proof (of Theorem 3.4). Let $b_{1,n}$ be a sequence of smooth drifts approximating b_1 . Then, using (2.6) and (3.17), we have

$$D_t^i X_s^{x,n} = e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \left(\int_t^s D_t^i b_2(u,\omega) e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} du + \sigma_i \right). \quad (3.26)$$

It follows from Corollary 2.4 that $(X_t^{x,n})_{n \geq 1}$ is relatively compact in $L^2(\Omega, P)$ and $\|D_s X_t^{x,n}\|_{L^2(P \otimes dt)}$ is uniformly bounded in n . Hence by [35, Lemma 1.2.3], $(D_s X_t^{x,n})_{n \geq 1}$ converges weakly to $D_s X_t^x$ in $L^2(P; \mathbb{R})$. Thus, in order to conclude, we need to show that

$$\left\{ e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \left(\int_t^s D_t^i b_2(u,\omega) e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} du + \sigma_i \right) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right\}_n$$

converges to

$$e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{X^x}(du,dz)} \left(\int_t^s D_t^i b_2(u,\omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du + \sigma_i \right) \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right)$$

in expectation for every $\varphi \in C_b^1([0, T], \mathbb{R}^d)$. We will only show that

$$\left\{ e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} du \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \right\}_n$$

converges to $e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{X^x}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right)$ in expectation. Using Girsanov theorem and the Cameron-Martin theorem, we have

$$\begin{aligned} L &= \left| E \left[\mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \left\{ e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} du \right. \right. \right. \\ &\quad \left. \left. \left. - e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{X^x}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \right\} \right] \right| \\ &= \left| E \left[\mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \left\{ e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} du \right. \right. \right. \\ &\quad \left. \left. - e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \right. \right. \\ &\quad \left. \left. + e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \right. \right. \\ &\quad \left. \left. \left. - e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{X^x}(du,dz)} \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \right\} \right] \right| \\ &= \left| E \left[\mathcal{E} \left(\int_0^T \dot{\varphi}_r dB_r \right) \left\{ e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} \right. \right. \right. \\ &\quad \times \int_t^s D_t^i b_2(u, \omega) \left(e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{X^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} \right) du \\ &\quad \left. \left. + \left(e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{X^{x,n}}(du,dz)} - e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{X^x}(du,dz)} \right) \int_t^s D_t^i b_2(u, \omega) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{X^x}(dr,dz)} du \right\} \right] \right| \\ &= \left| E \left[e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{\tilde{X}^{x,n}}(du,dz)} \right. \right. \\ &\quad \times \int_t^s D_t^i b_2(u, \omega + \varphi) \left(e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right) du \\ &\quad \left. \left. + \left(e^{-\int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{\tilde{X}^{x,n}}(du,dz)} - e^{-\int_t^s \int_{\mathbb{R}} b_1(u,z) L^{\tilde{X}^x}(du,dz)} \right) \int_t^s D_t^i b_2(u, \omega + \varphi) e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} du \right] \right| \\ &\leq I_{1,n} + I_{2,n}. \end{aligned} \tag{3.27}$$

Let us concentrate on $I_{1,n}$. Repeated use of Hölder inequality, Girsanov transform, the bound on $D_t^i b_2(u, \omega + \varphi)$ and the fact

that $|e^x - 1| \leq |x|(e^x + 1)$ gives

$$\begin{aligned}
I_{1,n} &\leq \left| E \left[e^{-2 \int_t^s \int_{\mathbb{R}} b_{1,n}(u,z) L^{\tilde{X}^{x,n}}(du,dz) \right]^{1/2} \times E \left[\left(\int_t^s (D_t^i b_2(u, \omega + \varphi))^2 du \right)^2 \right]^{1/4} \right. \\
&\quad \times E \left[\left(\int_t^s \left(e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right)^2 du \right)^2 \right]^{1/4} \Big| \\
&\leq CE \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) e^{-2 \int_t^s \int_{\mathbb{R}} b_{1,n}(r,z) L^{\|\sigma\| B_\sigma^x}(dr,dz)} \right]^{1/2} \\
&\quad \times \left(\int_t^s E \left[\left(e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right)^4 \right] du \right)^{1/4} \\
&\leq CE \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right)^2 \right]^{1/4} E \left[e^{-4 \int_t^s \int_{\mathbb{R}} b_{1,n}(r,z) L^{\|\sigma\| B_\sigma^x}(dr,dz)} \right]^{1/4} \\
&\quad \times \left(\int_t^s E \left[\left| e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right|^{1/2} \right]^{1/2} \right. \\
&\quad \times E \left[\left| e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right|^{15} \right]^{1/2} du \Big)^{1/4} \\
&\leq CE \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right)^2 \right]^{1/4} E \left[e^{-4 \int_t^s \int_{\mathbb{R}} b_{1,n}(r,z) L^{\|\sigma\| B_\sigma^x}(dr,dz)} \right]^{1/4} \\
&\quad \times \left(\int_t^s E \left[\left| e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} - e^{\int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right|^{1/2} \right]^{1/2} \right. \\
&\quad \times E \left[e^{15 \int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} + e^{15 \int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right]^{1/2} du \Big)^{1/4} \\
&\leq CE \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right)^2 \right]^{1/4} E \left[e^{-4 \int_t^s \int_{\mathbb{R}} b_{1,n}(r,z) L^{\|\sigma\| B_\sigma^x}(dr,dz)} \right]^{1/4} \\
&\quad \times \left(\int_t^s E \left[\left| \int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz) - \int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz) \right|^{1/2} \right]^{1/2} \right. \\
&\quad \times \left| e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz) + \int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} + 1 \right|^{1/2} \Big]^{1/2} \\
&\quad \times E \left[e^{15 \int_t^u \int_{\mathbb{R}} b_{1,n}(r,z) L^{\tilde{X}^{x,n}}(dr,dz)} + e^{15 \int_t^u \int_{\mathbb{R}} b_1(r,z) L^{\tilde{X}^x}(dr,dz)} \right]^{1/2} du \Big)^{1/4}
\end{aligned}$$

$$\begin{aligned}
&\leq CE \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right)^2 \right]^{1/4} E \left[e^{-4 \int_t^s \int_{\mathbb{R}} b_{1,n}(r, z) L^{\|\sigma\| B_\sigma^x} (dr, dz)} \right]^{1/4} \\
&\quad \times \left(\int_t^s E \left[\left| \int_t^u \int_{\mathbb{R}} b_{1,n}(r, z) L^{\tilde{X}^{x,n}} (dr, dz) - \int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\tilde{X}^x} (dr, dz) \right| \right] \right)^{1/4} \\
&\quad \times E \left[\left| e^{\int_t^u \int_{\mathbb{R}} b_{1,n}(r, z) L^{\tilde{X}^{x,n}} (dr, dz)} + e^{\int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\tilde{X}^x} (dr, dz)} - 1 \right| \right]^{1/4} \\
&\quad \times E \left[e^{15 \int_t^u \int_{\mathbb{R}} b_{1,n}(r, z) L^{\tilde{X}^{x,n}} (dr, dz)} + e^{15 \int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\tilde{X}^x} (dr, dz)} \right]^{1/2} du \Big)^{1/4}.
\end{aligned}$$

In the above $B^\sigma := \sum_{i=1}^n \frac{\sigma_i}{\|\sigma\|} B^i$ is a standard Brownian motion. Using Cauchy-Schwartz inequality, the Novikov's condition on b_2 and Beneš Theorem, the first term is finite for small time T . Using Lemma 3.8 and [10, Proposition 2.1.1] enables to conclude that the second term is bounded. Using once more Cauchy-Schwartz inequality, Girsanov transform and Lemma 3.8, one deduces that the fourth and fifth terms are bounded for small time T . Let us now focus on the third term. By Girsanov transform and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&E \left[\int_t^u \int_{\mathbb{R}} b_{1,n}(r, z) L^{\tilde{X}^{x,n}} (dr, dz) - \int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\tilde{X}^x} (dr, dz) \right] \\
&= E \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) \int_t^u \int_{\mathbb{R}} b_{1,n}(r, z) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right. \\
&\quad \left. - \mathcal{E} \left(\int_0^T \left\{ \tilde{u}(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) \int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right] \\
&= E \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) \int_t^u \int_{\mathbb{R}} (b_{1,n}(r, z) - b_1(r, z)) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right. \\
&\quad \left. + \left\{ \mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) - \mathcal{E} \left(\int_0^T \left\{ \tilde{u}(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) \right\} \int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right] \\
&\leq E \left[\mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right)^2 \right]^{1/2} E \left[\left(\int_t^u \int_{\mathbb{R}} (b_{1,n}(r, z) - b_1(r, z)) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right)^2 \right]^{1/2} \\
&\quad + E \left[\left\{ \mathcal{E} \left(\int_0^T \left\{ \tilde{u}_n(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) - \mathcal{E} \left(\int_0^T \left\{ \tilde{u}(r, \omega, x + \sigma \cdot B_r) + \dot{\varphi}_r \right\} dB_r \right) \right\}^2 \right]^{1/2} \\
&\quad \times E \left[\left(\int_t^u \int_{\mathbb{R}} b_1(r, z) L^{\|\sigma\| B_\sigma^x} (dr, dz) \right)^2 \right]^{1/2}. \tag{3.28}
\end{aligned}$$

Using Lemma 3.8, the first and the last terms on the right of (3.28) is bounded. Using (3.2), Minkowski, Doob maximal inequality and the dominated convergence theorem the second term converges to 0. Similar reasoning as in the proof of Lemma 2.5 enable to conclude that the third term converges to 0. Thus the result follows. \square

4. Stochastic differentiable flow for SDEs with random drifts

The aim of this section is to prove existence of a Sobolev differentiable stochastic flow for the SDE (1.1) with (non-Markovian) random drifts. Due to the additive decomposition assumption $b(t, \omega, x) = b_1(t, x) + b_2(t, \omega)$, the analysis of the flow turns out to be much easier than that of the Malliavin derivative considered above. Most of the result of this section will follow as adaptations of some arguments of Mohammed et al. [33].

Throughout this section, we denote $b_n := b_{1,n} + b_2$ where $b_{1,n} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, is the sequence of smooth functions with compact support approximating b_1 as introduced in the proof of Theorem 1.1, see Step 3.2.2. We further denote by $X^{s,x,n}$ the unique solution of the SDE associated to b_n with initial condition $X_s^{s,x,n} = x$. That is,

$$X_t^{s,x,n} = x + \int_s^t b_n(u, X_u^{s,x,n}, \omega) du + \sigma \cdot B_t.$$

We first prove differentiability of the solution $X^{s,x,n}$, and derive a uniform (in n) bound on its derivative.

Proposition 4.1. *Let $p \geq 1$ and $(s, x) \in [0, T] \times \mathbb{R}$ be fixed. If T is small enough, then for every $t \in [0, T]$, almost every trajectories of $x \mapsto X_t^{s,x,n}$ is differentiable and it holds*

$$E[|\partial_x X_t^{s,x,n}|^p] \leq \tilde{C}_{p,T,\sigma,\|\tilde{b}\|_\infty} \exp\left(C_{p,T,\sigma,\|\tilde{b}\|_\infty} |x|^2\right) \quad (4.1)$$

for some positive constants $\tilde{C}_{p,T,\sigma,\|\tilde{b}\|_\infty}$ and $C_{p,T,\sigma,\|\tilde{b}\|_\infty}$ depending on p, T, σ and $\|\tilde{b}\|_\infty$.

Proof. The differentiability of the trajectories of $x \mapsto X_t^{s,x,n}$ follows from the seminal work [26], from which we further obtain that $\partial_x X_u^{s,x,n} := \frac{\partial}{\partial x} X_u^{s,x,n}$ satisfies

$$\partial_x X_t^{s,x,n} = 1 + \int_s^t (b'_{1,n}(u, X_u^{s,x,n}) + b'_2(u, X_u^{s,x,n}, \omega)) \partial_x X_u^{s,x,n} du, \quad (4.2)$$

where $b'_{1,n} = \frac{\partial}{\partial x} b_{1,n}$. The solution of this (random) ODE can be explicitly given by

$$\partial_x X_t^{s,x,n} = \exp\left(\int_s^t (b'_{1,n}(u, X_u^{s,x,n}) + b'_2(u, X_u^{s,x,n}, \omega)) du\right).$$

Thus, Girsanov's theorem and successive applications of Hölder's inequality give (recall the definition of u_n given in (2.8))

$$\begin{aligned} E[|\partial_x X_t^{s,x,n}|^p] &\leq E\left[\exp\left(\int_s^t 2pb'_{1,n}(u, X_u^{s,x,n}) du\right)\right]^{1/2} E\left[\exp\left(\int_s^t 2pb'_2(u, X_u^{s,x,n}, \omega) du\right)\right]^{1/2} \\ &\leq E[e^{2pTM_2}]^{1/2} E\left[\mathcal{E}\left(\int_0^t u_n(u, x + \sigma \cdot B_u, \omega) dB_u\right) \exp\left(2p \int_s^t b'_{1,n}(u, x + \sigma \cdot B_u) du\right)\right]^{1/2} \\ &\leq CE\left[\mathcal{E}\left(4 \int_0^T u_n(u, x + \sigma \cdot B_u, \omega) dB_u\right)\right]^{1/8} E\left[\exp\left(6 \int_0^T |u_n(u, x + \sigma \cdot B_u, \omega)|^2 du\right)\right]^{1/8} \\ &\quad \times E\left[\exp\left(4p \int_s^t b'_{1,n}(u, x + \sigma \cdot B_u) du\right)\right]^{1/4} =: I_1^n \times I_2^n \times I_3^n. \end{aligned}$$

Since $b_{1,n}$ is of linear growth and b_2 square integrable, it holds $I_1^n = 1$. As shown in the proof of Lemma 2.3, if T is small enough, the sequence I_2^n is bounded, and as shown in (2.16), we have

$$\begin{aligned}
I_3^n &= \sum_{q=1}^{\infty} \frac{E \left[\left| \int_s^t (4p) b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^q \right]}{q!} \leq \sum_{q=1}^{\infty} \frac{(4p)^q E \left[\left| \int_s^t b'_{1,n}(u, x + \sigma \cdot B_u) du \right|^{2q} \right]^{\frac{1}{2}}}{q!} \\
&\leq \sum_{q=1}^{\infty} \frac{(4p)^q (C_\sigma)^q (1 + |x|^{2q})^{1/2} \|\tilde{b}\|_\infty^q |t - s|^q (q!)^{1/2}}{q!} \leq \sum_{q=1}^{\infty} \frac{2^q (4p)^q (C_\sigma)^q (1 + |x|^q) \|\tilde{b}\|_\infty^q |t - s|^q}{2^q (q!)^{1/2}} \\
&\leq \left(\sum_{q=1}^{\infty} \frac{2^{2q} (4p)^{2q} (C_\sigma)^{2q} (1 + |x|^q)^2 \|\tilde{b}\|_\infty^{2q} |t - s|^{2q}}{q!} \right)^{\frac{1}{2}} \left(\sum_{q=1}^{\infty} \frac{1}{2^{2q}} \right)^{\frac{1}{2}} \\
&\leq 2 \left(\exp \left(16p^2 (C_\sigma)^2 \|\tilde{b}\|_\infty^2 |t - s|^2 (1 + |x|^2) \right) \right)^{\frac{1}{2}} \\
&\leq 2 \exp \left(p^2 C_{\sigma, \|\tilde{b}\|_\infty^2} T (1 + |x|^2) \right).
\end{aligned}$$

This concludes the proof that there exist positive constants $\tilde{C}_{p,T,\sigma,\|\tilde{b}\|_\infty}$ and $C_{p,T,\sigma,\|\tilde{b}\|_\infty}$ depending on p, T, σ and $\|\tilde{b}\|_\infty$ such that

$$E [|\partial_x X_t^{s,x,n}|^p] \leq \tilde{C}_{p,T,\sigma,\|\tilde{b}\|_\infty} \exp \left(C_{p,T,\sigma,\|\tilde{b}\|_\infty} |x|^2 \right). \quad \square$$

Corollary 4.2. *Let $p \geq 2$. If T is small enough, and M_2 in (A2) is bounded, then it holds*

$$E [|\partial_x X_t^{s_1,x_1} - \partial_x X_t^{s_2,x_2}|^p] \leq C_p (\|\tilde{b}_1\|_\infty, |x|^2, T) \left(|s_1 - s_2|^{p/2} + |x_1 - x_2|^p \right)$$

for every $s_1, s_2, t \in [0, T]$, $x_1, x_2 \in \mathbb{R}$ and for some continuous function C_p increasing in each component.

In particular, for every $t \in [0, T]$ almost every trajectories¹ of $(s, x) \mapsto X_t^{s,x}$ is α -Hölder continuous with $\alpha < 1/2$ in s and $\alpha < 1$ in x .

Proof. Assume without loss of generality that $s_1 \leq s_2 < t$. For ease of notation, put $X^{i,n} := X^{s_i, x_i, n}$, $i = 1, 2$. Then, for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
X_t^{1,n} - X_t^{2,n} &= x_1 - x_2 + \int_{s_1}^t b_{1,n}(u, X_u^{1,n}) + b_2(u, X_u^{1,n}, \omega) du + \int_{s_1}^t \sigma \cdot dB_u \\
&\quad - \int_{s_2}^t b_{1,n}(u, X_u^{2,n}) + b_2(u, X_u^{2,n}, \omega) du - \int_{s_2}^t \sigma \cdot dB_u \\
&= x_1 - x_2 + \int_{s_1}^{s_2} b_{1,n}(u, X_u^{1,n}) + b_2(u, X_u^{1,n}, \omega) du + \int_{s_2}^t b_{1,n}(u, X_u^{1,n}) - b_{1,n}(u, X_u^{2,n}) du \\
&\quad + \int_{s_2}^t b_2(u, X_u^{1,n}, \omega) - b_2(u, X_u^{2,n}, \omega) du + \sigma \cdot B_{s_1} - \sigma \cdot B_{s_2}.
\end{aligned}$$

¹We use the convention $X_t^{s,x} = x$ whenever $t \leq s$.

Using Hölder continuity of the paths of Brownian motion and the mean-value theorem applied to $x \mapsto b_{1,n}(u, X_u^{s_1, x})$, this allows to obtain the following estimates:

$$\begin{aligned}
E[|X_t^{1,n} - X_t^{2,n}|^p] &\leq C_p \left\{ |x_1 - x_2|^p + |s_1 - s_2|^{p/2} + E \left[\left(\int_0^T |b_{1,n}(u, X_u^{1,n}) + b_2(u, X_u^{1,n}, \omega)|^2 du \right)^{p/2} \right] |s_1 - s_2|^{p/2} \right. \\
&\quad \left. + |x_1 - x_2|^p E \left[\left| \int_{s_2}^t \int_0^1 b'_{1,n}(u, X_u^{s_1, x_1 + \tau(x_2 - x_1), n}) \partial_x X_u^{s_1, x_1 + \tau(x_2 - x_1), n} d\tau du \right|^p \right] \right\} + CE \left[\int_{s_2}^t |X_u^{1,n} - X_u^{2,n}|^p du \right] \\
&\leq C_p \left\{ |x_1 - x_2|^p + |s_1 - s_2|^{p/2} + E \left[\left(\int_0^T |b_{1,n}(u, X_u^{1,n}) + b_2(u, X_u^{1,n}, \omega)|^2 du \right)^{p/2} \right] |s_1 - s_2|^{p/2} \right. \\
&\quad \left. + |x_1 - x_2|^p \int_0^1 E \left[\left| \partial_x X_t^{s_1, x_1 + \tau(x_2 - x_1), n} - \partial_x X_{s_2}^{s_1, x_1 + \tau(x_2 - x_1), n} \right|^p \right] d\tau \right\} + CE \left[\int_{s_2}^t |X_u^{1,n} - X_u^{2,n}|^p du \right] \\
&\leq C_p \left(|x_1 - x_2|^p + |s_1 - s_2|^{p/2} \right) \left\{ E \left[\left(\int_0^T |b_{1,n}(u, X_u^{1,n}) + b_2(u, X_u^{1,n}, \omega)|^2 du \right)^{p/2} \right] \right. \\
&\quad \left. + \sup_{t \in [0, T]; |x| \leq |x_1| + |x_2|} \sup_{n \in \mathbb{N}} E \left[\left| \partial_x X_t^{s_1, x, n} \right|^p \right] \right\} + C \int_{s_2}^t E[|X_u^{1,n} - X_u^{2,n}|^p] du. \tag{4.3}
\end{aligned}$$

Since $|b_{1,n}(t, x)| \leq \|\tilde{b}_1\|_\infty(1 + |x|)$ and b_2 has exponential moments (see (A1)) it follows by Jensen's inequality that

$$\begin{aligned}
E \left[\left(\int_0^T |b_{1,n}(u, X_u^{1,n}) + b_2(u, \omega)|^2 du \right)^{p/2} \right] &\leq C_p T \left(\|\tilde{b}_1\|_\infty E \left[\int_0^T 1 + |X_u^{1,n}|^{p/2} du \right] + b_2^{\exp} \right) \\
&\leq C_p T^2 \left(\|\tilde{b}_1\|_\infty (1 + \sup_n E[|X_t^{1,n}|^{p/2}]) + b_2^{\exp} \right).
\end{aligned}$$

Since by Lemma 2.5 it holds $\sup_n E[|X_t^{1,n}|^{p/2}] < \infty$, it follows by (4.3) (after application of Gronwall's inequality) and Proposition 4.1 that

$$E[|X_t^{1,n} - X_t^{2,n}|^p] \leq C_p(\|\tilde{b}_1\|_\infty, T) \left(|s_1 - s_2|^{p/2} + |x_1 - x_2|^p \right). \tag{4.4}$$

Let $i = 1, 2$. Since $(X_t^{s_i, x_i, n})$ converges weakly to the unique solution $X_t^{s_i, x_i}$ of the SDE (1.1) with drift b , (see Lemma 2.5 and Theorem 2.7) it follows by convexity and lower-semicontinuity of $K \mapsto E[|K|^p]$ that, taking the limit in (4.4) yields the desired result. \square

We conclude this section with the proof of Theorem 1.3.

Proof (of Theorem 1.3). Let $p \geq 2$. In order to show that $x \mapsto X^{s, x}$ is weakly differentiable, we start by showing that the

sequence $(\partial_x X_t^{s,x,n})_n$ is bounded in $L^2(\Omega, L^p(\mathbb{R}, w))$. In fact, it follows by Hölder's inequality and Proposition 4.1 that

$$\begin{aligned} E \left[\left(\int_{\mathbb{R}} |\partial_x X_t^{s,x,n}|^p w(x) dx \right)^{2/p} \right] &\leq E \left[\int_{\mathbb{R}} |\partial_x X_t^{s,x,n}|^p w(x) dx \right]^{2/p} \\ &= \left(\int_{\mathbb{R}} E [|\partial_x X_t^{s,x,n}|^p] w(x) dx \right)^{2/p} \\ &\leq \tilde{C}_{p,T,\sigma,\|\tilde{b}\|_\infty} \left(\int_{\mathbb{R}} \exp \left(C_{p,T,\sigma,\|\tilde{b}\|_\infty} |x|^2 \right) w(x) dx \right)^{2/p} < \infty. \end{aligned} \quad (4.5)$$

Thus, $(\partial_x X_t^{s,x,n})$ admits a weakly converging subsequence $(\partial_x X_t^{s,x,n_k})$ in $L^2(\Omega, L^p(\mathbb{R}, w))$ to a limit $Y_t^{s,x}$. Thus, for every $A \in \mathcal{F}$ and $f \in L^q(\mathbb{R}, w)$ (where q is the Sobolev conjugate of p), we have

$$\lim_{k \rightarrow \infty} E \left[1_A \int \frac{\partial}{\partial x} X_t^{n_k,x} f(x) w(x) dx \right] = E \left[1_A \int Y_t^{s,x} f(x) w(x) dx \right].$$

Choosing f such that $fw \in L^q(\mathbb{R}, dx)$, it follows that $Y_t^{s,x}$ is the weak derivative of $X_t^{s,x}$. By weak convergence and (4.1), it follows that

$$E \left[\left(\int_{\mathbb{R}} |\partial_x X_t^{s,x}|^p w(x) dx \right)^{2/p} \right] < \infty,$$

where $\partial_x X_t^{s,x}$ denotes the weak derivative of $X_t^{s,x}$.

It remains to show that

$$E \left[\left(\int_{\mathbb{R}} |X_t^{s,x}|^p w(x) dx \right)^{2/p} \right] < \infty. \quad (4.6)$$

By Jensen's inequality and the linear growth of b_1 , we have

$$\begin{aligned} &\int_{\mathbb{R}} |X_t^{s,x}|^p w(x) dx \\ &\leq C_p \left(\int_{\mathbb{R}} |x|^p w(x) dx + t^{p-1} \int_{\mathbb{R}} \int_0^t (1 + |X_u^{s,x}|^p) w(x) du dx + t^{p-1} \left\{ \int_0^t |b_2(u, X_u^{s,x}, \omega)|^p du + |\sigma \cdot B_t|^p \right\} \int_{\mathbb{R}} w(x) dx \right) \\ &\leq C_{p,T} \left(\int_{\mathbb{R}} |x|^p w(x) dx + \int_0^t \int_{\mathbb{R}} |X_u^{s,x}|^p w(x) dx du + \left\{ CTM_2^p + \sup_{t \in [0,T]} |\sigma \cdot B_t|^p \right\} \int_{\mathbb{R}} w(x) dx \right). \end{aligned}$$

By Gronwall's inequality, this implies

$$\int_{\mathbb{R}} |X_t^{s,x}|^p w(x) dx \leq C_{p,T} \left(\int_{\mathbb{R}} |x|^p w(x) dx + \left\{ T + CTM_2^p + \sup_{t \in [0,T]} |\sigma \cdot B_t|^p \right\} \int_{\mathbb{R}} w(x) dx \right) e^{T^2}.$$

Thus, it follows by Hölder's inequality that

$$\begin{aligned} E \left[\left(\int_{\mathbb{R}} |X_t^{s,x}|^p w(x) dx \right)^{1/p} \right] &\leq E \left[\int_{\mathbb{R}} |X_t^{s,x}|^p w(x) dx \right]^{1/p} \\ &\leq C_{p,T} \left(\int_{\mathbb{R}} |x|^p w(x) dx + \left\{ T + CTE[M_2^p] + |\sigma^p| E \left[\sup_{t \in [0,T]} |B_t|^p \right] \right\} \int_{\mathbb{R}} w(x) dx \right)^{1/p} < \infty. \end{aligned}$$

This completes the proof since by (1.4) M_2 has exponential moments. \square

A. Compactness criteria

The suggested construction of the strong solution for the SDE (1.6) is based on the subsequent relative compactness criteria from Malliavin calculus (see [37].)

Theorem A.1. *Let $\{(\Omega, \mathcal{A}, P); H\}$ be a Gaussian probability space, that is (Ω, \mathcal{A}, P) is a probability space and H a separable closed subspace of Gaussian random variables in $L^2(\Omega)$, which generate the σ -field \mathcal{A} . Denote by \mathbf{D} the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \dots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \quad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1} \mathbf{D}G\|_{L^2(\Omega; H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

The relative compactness criteria in this paper required the following result (see [37, Lemma 1]).

Lemma A.2. *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, 1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j < 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

The next compactness criteria comes from Theorem A.1 and Lemma A.2.

Corollary A.3. Let $\{X_n\}_{n \geq 1} \in \mathbf{D}_{1,2}$, be a sequence of \mathcal{F}_1 -measurable random variables such that there exist constants $\alpha > 0$ and $C > 0$ with

$$\sup_n E [|D_t X_n - D_{t'} X_n|^2] \leq C |t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$, and

$$\sup_n \sup_{0 \leq t \leq 1} E [|D_t X_n|^2] \leq C.$$

Then the sequence $\{X_n\}_{n \geq 1}$, is relatively compact in $L^2(\Omega)$.

B. An auxiliary result

The following key result generalises [7, Proposition 2.2.] to the case of function with spatial polynomial growth.

Proposition B.1. Let B be a d -dimensional Brownian motion starting from the origin and $b : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ a compactly supported smooth function such that $\|\tilde{b}(t, z)\| \leq k$ with $\tilde{b}(t, z) := \frac{b(t, z)}{1+|z|}$, $(t, z) \in [0, 1] \times \mathbb{R}$. Set $B_\sigma(t) = \sum_{i=1}^d \sigma_i B_i(t) \simeq N(0, t\|\sigma\|^2)$ for $\sigma \in \mathbb{R}^d$. Then there exists a constant C depending on σ such that for any even positive number n , we have

$$E \left[\left(\int_{t_0}^t b'(t, x + B_\sigma(t)) dt \right)^n \right] \leq C_{\sigma, k}^n (1 + |x|^n) \left(\frac{n}{2}\right)! (t - t_0)^{n/2}. \quad (\text{B.1})$$

Proof. As in [7], the proof is split into several parts.

Let $P_\sigma(t, z) = (2\pi t\|\sigma\|^2)^{-1/2} e^{-|z|^2/2t\|\sigma\|^2}$ be the Gaussian kernel, then using the joint distribution of $B(t_1), \dots, B(t_n)$, the left hand side of (B.1) can be written as

$$n! \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^n} \prod_{i=1}^n b'(t_i, x + z_i) P_\sigma(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n.$$

Define

$$J_n(t_0, t, x, z_0) := \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^n} \prod_{i=1}^n b'(t_i, x + z_i) P_\sigma(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n,$$

The proposition will be proved if we show that

$$|J_n(t_0, t, 0)| \leq C_p^n (t - t_0)^{n/2} (1 + |x|^n) / \Gamma(n/2 + 1).$$

Note that the above comes from Proposition 4.10. in Menoukeu-Pamen and Mohammed [29]. The result then follows. \square

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